Introduction to hydrodynamics – (Magneto-)hydrodynamical tools of non-linear flows

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Fluids: Plasma, gas, liquid

Gas kept at constant temperature in an enclosed container $\longrightarrow$ the pressure will be constant in the walls

The Continuum Hypothesis – Fluid dynamics

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Continuum hypothesis

The conditions that must be fulfilled so that the system can be considered as a fluid. The volume elements in a fluid may contain in fact a large number of particles (collectivity in number). The continuum hypothesis can also be characterized in terms of the mean free path of the fluid particles (collective momentum exchange). Therefore, in order that a given volume element be considered as continuous, the particle/molecular density must be sufficiently high, or the mean free path must be small with relation to some characteristic dimension R of the system, that is $\lambda \ll R$.

Example: Glass of Water

Let us consider a simple example of a fluid: a glass containing water. Considering the density of water in the usual conditions $\rho \simeq 1\ \mathrm{g/cm^3}$ and the molecular mass $m \simeq 18 m_H$, where $m_H = 1.67 \times 10^{-24}\ \mathrm{g}$ is the mass of a hydrogen atom, the number density of the water molecules in the glass is

$$n \simeq \frac{\rho}{m} \simeq \frac{\rho}{18 m_H} \simeq 3 \times 10^{22}\ \mathrm{cm}^{-3}.$$

In order of magnitude, the molecular mean free path can be approximated by the average separation of the molecules,

$$\lambda \sim n^{-1/3} \simeq 3 \times 10^{-8}\ \mathrm{cm}.$$

A characteristic dimension of the system (the glass) is the average radius, $R \simeq 5\ \mathrm{cm}$. Therefore, we have $\lambda \ll R$, so that the continuum hypothesis is fulfilled.

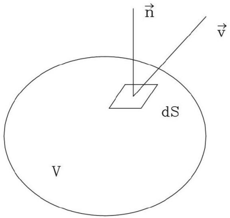
The state of an ideal fluid can be described by the velocity distribution of the fluid, characterized by the velocity vector $\vec{v}(\mathrm{x}, \mathrm{y}, z, \mathrm{t})$ and by two thermodynamic variables, such as the pressure $\mathrm{P}(\mathrm{x}, \mathrm{y}, z, \mathrm{t})$ and the density $\rho(\mathrm{x}, \mathrm{y}, z, \mathrm{t})$

Let us consider a volume V of the fluid. The total fluid mass contained in this volume is

$$\int_V \rho dV$$

The amount of matter that crosses dS per unit time is

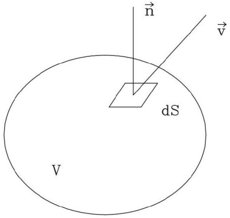
$$\rho \vec{v} \cdot \vec{n} dS$$



The total fluid mass that leaves volume V per unit time is

$$\oint \rho \vec{v} \cdot \vec{n} dS$$

The integral extends through the whole surface around volume V



On the other hand, the fluid mass decrease per unit time is

$$-\frac{\partial}{\partial t} \int_V \rho dV$$

In order to have mass conservation

$$\frac{\partial}{\partial t}\int_V \rho dV = -\oint \rho\vec{v}\cdot\vec{n}dS.$$

According to the divergence theorem, if $\vec{A}$ is a vector,

$$\int_V \vec{\nabla}\cdot\vec{A}dV = \oint \vec{A}\cdot\vec{n}dS,$$

so that

$$\oint \rho\vec{v}\cdot\vec{n}dS = \int_V \vec{\nabla}\cdot(\rho\vec{v})dV.$$

Therefore

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_V \vec{\nabla}\cdot(\rho\vec{v})dV$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right] dV = 0.$$

The Continuity Equation is thus $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$.

Example: Incompressible Fluids $\rho = \text{constant} \longrightarrow \quad \vec{\nabla} \cdot \vec{v} = \vec{0}.$

WARNING! and HINT!

An incompressible Fluid is different than an incompressible dynamics! The operator $\vec{\nabla} \cdot \vec{v} \equiv \mathrm{div}\, \vec{v}$!

Example: Steady State

$$\partial\rho/\partial t = 0 \longrightarrow \quad \vec{\nabla} \cdot (\rho\vec{v}) = 0$$

Example: Steady State and incompressible dynamics

$$\partial\rho/\partial t = 0 \quad \wedge \quad \vec{\nabla} \cdot \vec{v} = 0 \longrightarrow \quad \vec{\nabla} \cdot (\rho\vec{v}) = \rho\vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla}\rho = \vec{v} \cdot \vec{\nabla}\rho = 0$$

The Mass Flux $\vec{j} = \rho\vec{v}$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

Let us consider again a volume V in the fluid. The total force acting on this volume due to the interactions with the remaining fluid particles is

$$-\oint P\,\vec{n}\cdot dS \tag{1}$$

where the negative sign is due to the fact that the force acts on the considered element. Applying the 'gradient theorem' ('Gauss-Stokes') to a scalar quantity A, we have

$$\oint A\,\vec{n}\cdot dS = \int_V \vec{\nabla} A\,dV \quad \longrightarrow \quad -\oint P\,\vec{n}\cdot dS = \int_V \vec{\nabla} P\,dV \tag{2}$$

...the relaxed, flowing stream, without force? No the partial derivative neglects curvature and acceleration due to stationary displacements, thus total time derivative is necessary

$$\begin{aligned} \text{force} &= \rho \times \text{acceleration} = \rho \frac{d\vec{v}}{dt} = \rho \left(\sum_{k,l=1}^{k,l=3} v_k \frac{\partial v_l}{\partial x_k} \vec{e}_l + \frac{\partial v_k}{\partial t} \vec{e}_k \right) \\ &= \rho \left(\left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} + \frac{\partial \vec{v}}{\partial t} \right) \end{aligned} \tag{3}$$

Basic ideal hydrodynamical (HD) equations

Mass continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0\,, \tag{4}$$

Momentum equation (Euler equation)

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} \right) \quad = \quad -\vec{\nabla} p - \rho \vec{g} + \vec{f}\,, \tag{5}$$

Energy equation (energy conservation)

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{S} \quad = \quad \rho \vec{g} \cdot \vec{v} \tag{6}$$

$$\vec{S} \quad = \quad (U + p)\vec{v} \quad \wedge \quad U = \frac{p}{\gamma - 1} + \frac{1}{2}\rho \vec{v}^2 \tag{7}$$

Basic ideal MHD equations

Mass continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathrm{v}}) \quad = \quad 0\,, \tag{8}$$

Momentum equation (Euler equation)

$$\rho \left(\frac{\partial \vec{\mathrm{v}}}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla} \right) \vec{\mathrm{v}} \right) \quad = \quad -\vec{\nabla} p + \frac{1}{\mu_0} \left(\vec{\nabla} \times \vec{B} \right) \times \vec{B} - \rho \vec{g}\,, \tag{9}$$

Energy equation (Energy conservation)

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{S} = \rho \vec{g} \cdot \vec{v} \tag{10}$$

$$\vec{S} = (U + p + \frac{\vec{B}^2}{2\mu_0})\vec{v} - \frac{\vec{v} \cdot \vec{B}}{\mu_0} \vec{B}\,, U = \frac{p}{\gamma - 1} + \frac{\rho \vec{v}^2}{2} + \frac{\vec{B}^2}{2\mu_0} \tag{11}$$

No magnetic monopoles & Ampère's law (initial condition/constraint & quasi-stationary current)

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \wedge \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \tag{12}$$

....but although one fluid theory, we have two fluids, the relic of the electron momentum equation and Faraday's equation (induction equation), as relic of Maxwell equations

...the ideal Ohm's law & Faraday's law

$$\vec{E} + \vec{v} \times \vec{B} = \vec{0} \quad \wedge \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) = \vec{0} \qquad (13)$$

The last equation is somestimes also called 'Induction equation of ideal MHD' .

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{14}$$

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \times \vec{B}\right) \cdot \vec{C} \tag{15}$$

$$= \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \left(\vec{B} \times \vec{C}\right) \cdot \vec{A} \tag{16}$$

$$= \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) \tag{17}$$

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{C} \times \vec{B}\right) \times \vec{A} = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) \tag{18}$$

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f \tag{19}$$

$$\vec{\nabla} \cdot (f\vec{A}) = f\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla}f \tag{20}$$

$$\vec{\nabla} \times (f\vec{A}) = f\vec{\nabla} \times \vec{A} + \vec{\nabla}f \times \vec{A} \tag{21}$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \tag{22}$$

Useful relations for vector equations: vector algebra/analysis identities

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot B) + (\vec{B} \cdot \vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla} \cdot \vec{A}) - (\vec{A} \cdot \vec{\nabla})\vec{B} \tag{23}$$

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A} \tag{24}$$

$$\Delta f = \vec{\nabla}^2 f = \vec{\nabla} \cdot \vec{\nabla} f \tag{25}$$

$$\Delta \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \times \vec{\nabla} \times \vec{A} \tag{26}$$

$$\vec{\nabla} \times \vec{\nabla} f = \vec{0} \tag{27}$$

$$\vec{\nabla} \cdot \vec{r} = 3\,, \quad \vec{r} := (x, y, z) \tag{28}$$

Example: Rotational flow and expansion of ‘curl’ respectively ’rot’

$$
\begin{aligned}
\vec{\omega} &= (0, 0, \omega = \text{constant}) := 0\vec{e}_x + 0\vec{e}_y + \omega\vec{e}_z \quad \wedge \quad \vec{r} := (x, y, z) \\
\vec{v} : &= \vec{\omega} \times \vec{r} \\
\text{rot}\,\vec{v} &= \vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r}) \\
&= \vec{\omega}\left(\vec{\nabla} \cdot \vec{r}\right) - \vec{r}\left(\vec{\nabla} \cdot \vec{\omega}\right) + \left(\vec{r} \cdot \vec{\nabla}\right)\vec{\omega} - \left(\vec{\omega} \cdot \vec{\nabla}\right)\vec{r} \\
&\overset{general}{=} 2\vec{\omega} \\
&\overset{here}{=} 2\omega\,\vec{e}_z
\end{aligned} \tag{29}
$$

Implication: ω is like a rotational frequency, reflecting the rotation strength of the flow around a centre!

- approach for divergence and non-divergence free fields

The wish to get a vortex-free field (no or zero vorticity) the following potential field representation is useful, namely

$$\begin{aligned} & \vec{v} = \vec{\nabla}\varphi \\ \Rightarrow \qquad & \vec{\Omega} := \vec{\nabla} \times \vec{v} = \vec{0} \,. \end{aligned} \tag{30}$$

To manipulate, i.e. to extend, the representation Eq. (30), it is not enough to add a second gradient of a potential, let us say $\vec{\nabla}\beta$, because the 'curl' operator leaves the vorticity $\vec{\Omega}$ unchanged, i.e. equal to zero.

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A way out is for example to introduce a scalar function, let us say α as multiplicator in front of the expression $\vec{\nabla}\beta$, writing

$$\vec{v} = \vec{\nabla}\varphi + \alpha\,\vec{\nabla}\beta\,, \tag{31}$$

which is often called Clebsch representation.

The second term is a non-linear gradient term, which is not necessarily a full gradient, indeed is *usually* in the non-trivial case never a full gradient! The second part of the sum in Eq. (31) is a complete gradient, if, and only if, $\beta = \beta(\alpha)$, $\alpha = \alpha(\beta)$ respectively, at least locally, such that

$$
\begin{aligned}
& \vec{\nabla} \times \left(\alpha \vec{\nabla} \beta(\alpha)\right) = \vec{\nabla}\alpha \times \left(\beta'(\alpha) \vec{\nabla}\alpha\right) = \vec{0} \\
\Leftrightarrow \qquad & \exists\ G = \int \alpha\, d\beta = \int \alpha\beta'(\alpha)\, d\alpha\,,
\end{aligned}
\tag{32}
$$

with $\vec{\nabla} G = \alpha \vec{\nabla}\beta$, and $\beta'(\alpha)$ as derivative of β with respect to α. This is the integrability recipe Eq. (32) for completing a gradient like in Eq. (31) .

Returning to the non-trivial non-linear potential representation, i.e. Eq. (31), its vorticity is given by

$$\begin{aligned} \vec{\Omega} &= \vec{\nabla} \times \left(\vec{\nabla}\varphi + \alpha\,\vec{\nabla}\beta\right) = \vec{\nabla} \times \left(\alpha\,\vec{\nabla}\beta\right) = \vec{\nabla}\alpha \times \vec{\nabla}\beta + \alpha\,\vec{\nabla} \times \vec{\nabla}\beta \\ \Rightarrow \vec{\Omega} &= \vec{\nabla}\alpha \times \vec{\nabla}\beta\,. \end{aligned} \tag{33}$$

As the vorticity is a 'curl', it is automatically divergence free, i.e. $\vec{\nabla} \cdot \vec{\Omega} = 0$. What is a vector potential representation?

$$\vec{v} = v_x \vec{\nabla} x + v_y \vec{\nabla} y + v_z \vec{\nabla} z\,. \tag{34}$$

This reflects a 1-D character??? of Eq. (30) because one has only one coordinate, although the potential function ϕ depends basically on all three space coordinates.

Warning!

Potentials, representing the fields can be regarded as coordinates, but one should be aware that they may not be unique with respect to the usual unique spatial coordinates, which are diffeomorphisms/manifolds of 2-or 3-dimensional space (points), see e.g. Grad & Rubin (1958), Arnold (1980).

From equations like, e.g., Eq. (33), one can derive conservation laws, i.e. (instead of the vorticity $\vec{\Omega}$, we regard here the vector field $\vec{V} = \vec{\nabla}\alpha \times \vec{\nabla}\beta$)

$$\vec{V} \cdot \vec{\nabla}\alpha = \left(\vec{\nabla}\alpha \times \vec{\nabla}\beta\right) \cdot \vec{\nabla}\alpha = -\left(\vec{\nabla}\alpha \times \vec{\nabla}\alpha\right) \cdot \vec{\nabla}\beta = 0 \quad \wedge \quad \vec{V} \cdot \vec{\nabla}\beta = 0 \quad (35)$$

The directional derivative with respect to a specific vector field implies the conservation of a value along its trajectories, i.e. along the field lines of the vector field, here a consequence of their Euler potential representation. What conservation laws are valid if the Euler potentials are not explicitly given, but instead only one, $\vec{V}_1$ or two vector fields, $\vec{V}_1$ and $\vec{V}_2$, or one knows only one vector field, but two integrals, eventually not matching as Euler potential, with

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$$\vec{V} \cdot \vec{\nabla} I = 0 \quad \text{or} \quad \vec{V}_1 \cdot \vec{\nabla} I = 0 \quad \wedge \quad \vec{V}_2 \cdot \vec{\nabla} I = 0\,, \qquad (36)$$

with *one* known first integral I, are given.

'Naive ' implication of Eq. (36), but also an unmatched Eq. (35), where $\vec{V}$ is not explicitely given

$$\vec{V}_1 \times \vec{V}_2 \parallel \vec{\nabla} I \;\Leftrightarrow\; \vec{V}_1 \times \vec{V}_2 = \tilde{\lambda}\vec{\nabla} I \quad \Rightarrow \quad \vec{V}_1 = \lambda\, \vec{V}_2 \times \vec{\nabla} I + \mu \vec{V}_2 \qquad (37)$$

$$V \cdot \vec{\nabla}\tilde{\alpha} = 0 \quad \wedge \quad V \cdot \vec{\nabla}\tilde{\beta} = 0 \qquad \vec{V} = \lambda \vec{\nabla}\tilde{\alpha} \times \vec{\nabla}\tilde{\beta} \qquad (38)$$

We will now consider a specific example for the above formulated, general conservation theorem Eq.(36), namely a conservation law following from the Euler equation, with the help of the Weber-Grassmann identity

$$\frac{1}{2}\,\vec{\nabla}\left(\vec{V}\right)^2 = \vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\,. \tag{39}$$

The Euler equation Eq.(40), i.e. the first order momentum or force equation, can therefore be re-written according the Weber-Grassmann identity as

$$\frac{\vec{\nabla}p}{\rho}+\left(\vec{\mathrm{v}}\cdot\vec{\nabla}\right)\vec{\mathrm{v}} = \vec{0}\,\left[-\vec{\nabla}\phi\right] \tag{40}$$

$$\Rightarrow \qquad \frac{\vec{\nabla}p}{\rho}+\frac{1}{2}\,\vec{\nabla}\vec{\mathrm{v}}^2 = \vec{\mathrm{v}}\times\vec{\Omega}\,. \tag{41}$$

The conclusion is that the expression

$$\mathit{\Pi} := p+\frac{\rho}{2}\,\vec{\mathrm{v}}^2\,, \tag{42}$$

often called total pressure or Bernoulli pressure, is constant in a domain, where the density is constant, if the vector product

$$\vec{\mathrm{v}}\times\vec{\Omega} = \vec{0}\,, \tag{43}$$

often called Beltrami equation, i.e. vanishes everywhere in the domain. The conservation of the total pressure $\mathit{\Pi}$ is in some sense energy conservation.

Assuming that the curl of the velocity field vanishes, i.e. $\vec{v}$ is a potential field with $\vec{v} = \vec{\nabla}\varphi$,

$$\vec{\Omega} = \vec{\nabla} \times \vec{v} = \vec{0}\,, \tag{44}$$

is a possibility to fulfill Eq. (43). To fulfill the mass continuity or mass conservation equation with $\rho =$constant, we can write

$$\vec{\nabla} \cdot (\rho\vec{v}) = 0 \;\Rightarrow \vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \vec{\nabla}\varphi = \Delta\varphi = 0\,, \tag{45}$$

i.e. one has to solve the Laplace equation. The velocity fields which are everywhere (*only*) parallel to its own vortices

$$\vec{\Omega} \times \vec{v} = \left(\vec{\nabla} \times \vec{v}\right) \times \vec{v} = \vec{0} \tag{46}$$

are called *Beltrami fields*.

For incompressible fluids $\rho =$constant Beltrami- or potential fields guarantee therefore the conservation of the total pressure Π, which is sometimes, although its very specific constraints, called *Bernoulli's theorem*. The problem of an Euler equation can therefore be solved, by solving either (i) the potential equation, i.e. the Laplace equation Eq. (45) (as far as the velocity field is definitely vortex-free), or solving the Beltrami equation (ii) Eq.(46), and then, for both cases, the pressure can be calculated from the total pressure Eq. (42).

As for the incompressible flows $\vec{v} \cdot \vec{\nabla} \rho = 0$, i.e. also $\vec{v} \cdot \vec{\nabla} f(\rho) = 0$, the basic Euler equation Eq.(40) can be manipulated by scalar multiplication with $\vec{v}$, such that

$$\vec{v} \cdot \vec{\nabla} \left(\frac{p}{\rho} + \frac{1}{2} \vec{v}^2 \right) = \vec{v} \cdot \left(\vec{v} \times \vec{\Omega} \right) = 0 \ \Rightarrow \ \vec{v} \cdot \vec{\nabla} \left(p + \frac{\rho}{2} \vec{v}^2 \right) = 0 \,, \tag{47}$$

$$\frac{\vec{\Omega} \cdot \vec{\nabla} p}{\rho} + \frac{1}{2} \vec{\nabla} \vec{v}^2 = \vec{\Omega} \cdot \left(\vec{v} \times \vec{\Omega} \right) = 0 \,, \tag{48}$$

$$\Rightarrow \qquad \vec{\Omega} \cdot \vec{\nabla} \left(p + \frac{\rho}{2} \vec{v}^2 \right) - \frac{\vec{v}^2}{2} \vec{\Omega} \cdot \vec{\nabla} \rho = 0 \tag{49}$$

respectively. The first conservation law Eq. (47) is usually that what is called *Bernoulli's theorem* or *Bernoulli's equation*. *Remark: Here no restrictions to potential or Beltrami fields are necessary, only the restriction that the mass density must be constant along streamlines.*

Conditions and definitions for incompressible flows

$$\vec{\nabla} \cdot \vec{v} = 0 \quad \Rightarrow \quad \vec{v} \cdot \vec{\nabla}\rho = 0 \quad \wedge \quad \vec{v} \perp \vec{\nabla}\rho \tag{50}$$

$$\vec{w} := \sqrt{\rho}\vec{v} \quad \text{streaming–vector} \tag{51}$$

$$\Rightarrow \qquad \vec{w} \cdot \vec{\nabla}\rho = 0 \quad \wedge \quad \vec{\nabla} \cdot \vec{w} = 0 \tag{52}$$

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Implications from stationary momentum equation of HD by applying the Weber transformation or Weber-Grassmann identity

$$-\frac{1}{\rho}\vec{\nabla}p = (\vec{v} \cdot \vec{\nabla})\vec{v} \quad \Leftrightarrow \quad \vec{\nabla}\Pi = \vec{w} \times (\vec{\nabla} \times \vec{w}), \tag{54}$$

Basic assumptions and conclusions for 3D incompressible flows

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Implications of the resulting momentum equation in 3D

$$\vec{\nabla}\Pi=\vec{w}\times\left(\vec{\nabla}\times\vec{w}\right) \;\Rightarrow\; \vec{w}\cdot\vec{\nabla}\Pi=0 \;\wedge\; \left(\vec{\nabla}\times\vec{w}\right)\cdot\vec{\nabla}\Pi=0 \tag{55}$$

Stationary HD – MHS/analogy: Lagrange-Stokes equation/Grad–Shafranov equation & Bernoulli equation

Implications from stationary momentum equation of HD by applying the Weber-Grassmann identity in case of incompressibility ($\vec{\nabla} \cdot \vec{v} = 0$) and $\vec{w} = \sqrt{\rho}\,\vec{v}$ with $\vec{\nabla} \cdot \vec{w} = 0$...

$$-\frac{1}{\rho}\vec{\nabla} p = (\vec{v} \cdot \vec{\nabla})\vec{v} \;\Leftrightarrow\; \vec{\nabla} \mathit{\Pi} = \vec{w} \times (\vec{\nabla} \times \vec{w}) \,, \tag{56}$$

where $\mathit{\Pi} = p + \rho \vec{v}^2/2 = p + \vec{w}^2/2$ is the total or Bernoulli-pressure.

Implications from stationary momentum equation of HD by applying the Weber-Grassmann identity in case of incompressibility ($\vec{\nabla} \cdot \vec{v} = 0$) and $\vec{w} = \sqrt{\rho}\, \vec{v}$ with $\vec{\nabla} \cdot \vec{w} = 0$...

$$-\frac{1}{\rho} \vec{\nabla} p = (\vec{v} \cdot \vec{\nabla}) \vec{v} \;\Leftrightarrow\; \vec{\nabla} \mathit{\Pi} = \vec{w} \times (\vec{\nabla} \times \vec{w})\,, \tag{56}$$

where $\mathit{\Pi} = p + \rho \vec{v}^2/2 = p + \vec{w}^2/2$ is the total or Bernoulli-pressure.

With *pure* 2D approach

$$\vec{w} = \vec{\nabla} \times (\psi \vec{e}_z) = \vec{\nabla} \psi \times \vec{e}_z \;\Rightarrow\; \Delta \psi = \frac{d\mathit{\Pi}}{d\psi} \tag{57}$$

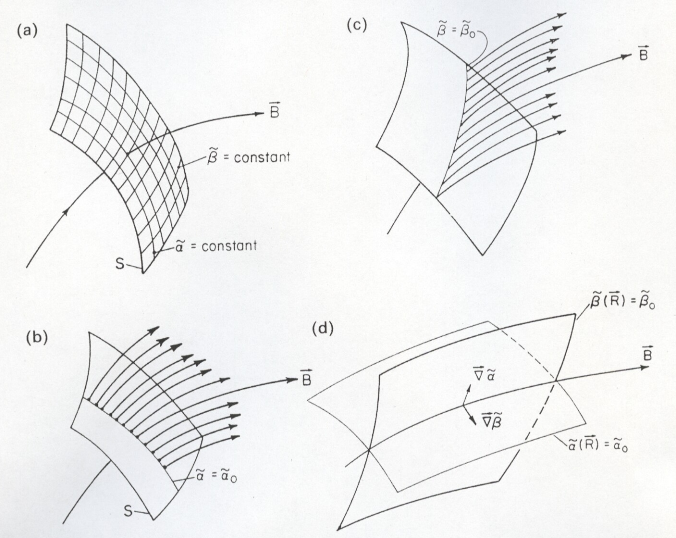
Most important properties: Symmetries $\rightarrow$ conservation laws

$\mathit{\Pi}$ is conserved along field lines, $\mathit{\Pi}$ is so called ‘first integral’ , i.e.
$\vec{w} = \vec{\nabla} \alpha \times \vec{\nabla} \beta \quad \Rightarrow \quad \mathit{\Pi} = \mathit{\Pi}(\alpha, \beta)$

SHD equations with Hamiltonian structure

$$\begin{aligned} \vec{\nabla}\beta \cdot \vec{\nabla} \times (\vec{\nabla}\alpha \times \vec{\nabla}\beta) &= -\frac{\partial \Pi}{\partial \alpha} \\ \vec{\nabla}\alpha \cdot \vec{\nabla} \times (\vec{\nabla}\alpha \times \vec{\nabla}\beta) &= \frac{\partial \Pi}{\partial \beta} \,. \end{aligned} \tag{58}$$

Hamiltonian structure, conserved values, potential representation similar to GS-theory


(a)
$\vec{B}$
$\tilde{\beta}$ = constant
$\tilde{\alpha}$ = constant
S
(b)
$\vec{B}$
$\tilde{\alpha} = \tilde{\alpha}_0$
S
(c)
$\tilde{\beta} = \tilde{\beta}_0$
$\vec{B}$
(d)
$\tilde{\beta}(\vec{R}) = \tilde{\beta}_0$
$\vec{B}$
$\vec{\nabla}\tilde{\alpha}$
$\vec{\nabla}\tilde{\beta}$
$\tilde{\alpha}(\vec{R}) = \tilde{\alpha}_0$

As 2D example: MHS without gravitation and flows leads to Grad-Shafranov equation

Vector potential, assume $\alpha = A(x,y)$ and $\beta = z(+h(x,y))$

$$\begin{aligned} \vec{B}_S(x,y) &= \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(A\vec{e}_z\right) \; (+\vec{\nabla}A \times \vec{\nabla}h) \\ &= \vec{\nabla}A \times \vec{\nabla}z \; (+\vec{\nabla}A \times \vec{\nabla}h) \end{aligned} \tag{59}$$

$$\begin{aligned} &\vec{B}_S = \vec{\nabla}A \times \vec{e}_z + B_{zS}\vec{e}_z \;\Rightarrow\; \vec{B}_S \cdot \vec{\nabla}A = 0 \\ &\Rightarrow\; A(x,y) = \mathrm{const\ are}\; \textit{fieldlines}(-\textit{projections}). \end{aligned} \tag{60}$$

Grad–Shafranov equation (*Grad, 1960*, *Shafranov, 1958*)

$$\vec{\nabla}p_S = \vec{j}_s \times \vec{B}_s \;\Rightarrow\; \frac{d}{dA}\left(p_S + B_{zS}^2/2\right) = -\frac{1}{\mu_0}\Delta A \tag{61}$$

Vector potential, assume $\alpha = \psi(x, y)$ and $\beta = z(+h(x, y))$

$$\begin{aligned} \vec{w}\,(x,y) &= \vec{\nabla} \times \vec{\psi} = \vec{\nabla} \times (\psi \vec{e}_z)\ (+\vec{\nabla}\psi \times \vec{\nabla} h) \\ &= \vec{\nabla}\psi \times \vec{\nabla} z\ (+\vec{\nabla}\psi \times \vec{\nabla} h) \end{aligned} \tag{62}$$

$$\vec{w} = \vec{\nabla}\psi \times \vec{e}_z + w_{zS}\vec{e}_z \ \Rightarrow\ \vec{w} \cdot \vec{\nabla}\psi = 0$$
$$\Rightarrow\ \psi(x,y) = \mathrm{const\ are}\ \textit{streamlines}(-\textit{projections}). \tag{63}$$

Lagrange–Stokes equation (1781, 1848)

$$\vec{\nabla}\Pi = \vec{w} \times \vec{\Omega}\ \Rightarrow\ \frac{d}{d\psi}\left(\Pi - w_z^2/2\right) = \Delta\psi \tag{64}$$