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Physics of Extreme Massive Stars
Marie-Curie-RISE project funded by the European Union

Radiative Transfer

Basic Equations

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Introduction

The radiative transfer theory or radiation transport theory describes

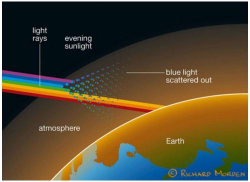


Figure 1: Earth's atmosphere

Figure 2: The Pleiades - Light scattering.

the energy transfer of electromagnetic waves (or photons) propagating through a medium that can:

- Absorb radiation
- Emit radiation
- Scatter radiation

The radiative transfer theory involves many research fields:

- theoretical astrophysics
- remote sensing
- cosmology
- physics (theory of neutron transport, flows in hyper-compressed layers, nuclear reactors)
- engineering (rocket engines, plasma generators and solar sailing)

At the beginning of 1880, the transport of energy by radiation through a medium became important to explain the distorted spectrum of the Sun when compared with a black body model.

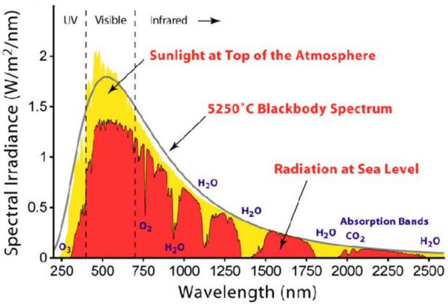


Figure 3: Comparison between a black body (5700 K) and the sun spectrum.

Extracted from http://www.globalwarmingart.com/wiki/File:Solar-Spectrum-png

Milestones: Radiative Transfer Theory

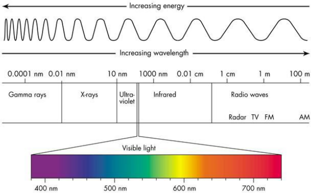
Stellar Astrophysics:

- The 1920's → Theory of monochromatic scattering (Eddington, Milne, and others)
- 1940 - 1960 → accurate approximate solutions to the basic radiative transfer equations (Ambartsumian, Sobolev, Chandrasekhar)
- 1960 -1990 Improved solutions for static and moving media (Feautier, Mihalas, Rybicki, Simmoneau)
- Today → accessible computer programs
  - for plane-parallel atmospheric layers (ATLAS12, SYNTHE, TLUSTY, SYNSPEC, PHOENIX, etc)
  - form spherical moving media (e.g., CMFGEN; FASTWIND, MCRT, PoWR, APPEL, IIM, among others).

Description of the Radiation Field

Classical Theory

Electromagnetic radiation is defined by its amplitude and colour (ν)



The specific intensity $I_\nu(\bar{r}, \bar{n}, t)$ represents

$$\delta E_\nu \;=\; I_\nu(\bar{r}, \bar{n}, t)\, d\nu\; ds\, \cos\theta\, dt\, d\bar{\Omega}_{\bar{n}}.$$

Quantum Theory

Photons of $E = h\nu$ and $\bar{p} = (h\nu/c)\,\bar{n}$.

$I_\nu(\bar{r}, \bar{n}, t) \propto\; h\nu\,\delta\mathcal{N}$

$\mathcal{N}(\bar{r}, \bar{n}, t)$: occupation number of photons per unit volume phase space at time t.

$$\delta\mathcal{N}(\bar{r}, \bar{p}, t) = \Phi(\bar{r}, \bar{p}, t)\, d\bar{r}^3\, d\bar{p}^3$$

$\Phi(\bar{r}, \bar{p}, t)$ is the phase-space density.

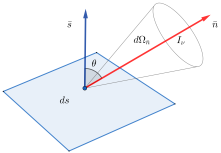


Figure 4: Radiation beam within the solid angle $d\Omega_{\bar{n}}$ in the direction $\bar{n}$.

$$\delta E = h\nu\, \delta\mathcal{N} = h\nu\, \Phi(\bar{r}, \bar{p}, t)\, d\bar{r}^3\, d\bar{p}^3$$

$d\bar{p}^3 = dp_x\, dp_y\, dp_z$ (cartesian coordinates)

In spherical coordinates:

$$\begin{aligned} dp_x &= p^2 dp\, \sin\theta \\ dp_y &= p^2 dp\, \sin\theta \cos\phi\;\, d\theta\, d\phi \\ dp_z &= p^2 dp\, \cos\theta \cos\phi\, d\theta\, d\phi \end{aligned}$$

$$dp^3 = p^2\, dp\, d\Omega_{\bar{n}} \quad \text{and} \quad p^2\, dp = \frac{h^3\, \nu^2}{c^3}\, d\nu$$

$$\delta E = (h\nu)\, \Phi(\bar{r}, \bar{n}, t)\, ds\, \cos\theta\, c\, dt\, p^2\, dp\, d\Omega_{\bar{n}}$$

$$\delta E = (h\nu)\, \Phi(\bar{r}, \bar{n}, t)\, (\tfrac{h^3\, \nu^2}{c^2})\, ds\, \cos\theta\, dt\, d\nu\, d\Omega_{\bar{n}}$$

The specific intensity is a distribution function,

$$\boxed{I_\nu(\bar{r}, \bar{n}, t) = \frac{h^4\, \nu^3}{c^2}\, \Phi_\nu(\bar{r}, \bar{n}, t).}$$

Recovering the classical definition
$\delta E_\nu = I_\nu(\bar{r}, \bar{n}, t)\, d\nu\;\, ds\, \cos\theta\, dt\, d\bar{\Omega}_{\bar{n}}$.

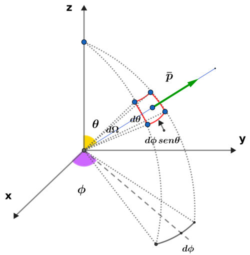
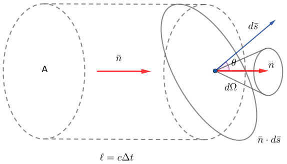


Figure 5: Spherical coordinates, where $d\Omega = sen\,\theta\, d\theta\, d\phi$.
θ: colatitude angle
ϕ: azimuthal angle



The Kinetic Equation

The evolution of the function $\Phi(\bar{r}, \bar{p}, t)$ $\Rightarrow$ a change in $\mathcal{N}$ and must satisfy a typical kinetic equation,

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t}_{coll} + S.$$

The total derivative of the distribution function

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \sum_i \frac{\partial\Phi}{\partial r_i}\frac{\partial r_i}{\partial t} + \sum_i \frac{\partial\Phi}{\partial p_i}\frac{\partial p_i}{\partial t},$$

where $\frac{\partial r_i}{\partial t} = v_i$ is the i-component of the velocity
$\frac{\partial p_i}{\partial t} = \mathfrak{f}_i$ is the i-component of the force exerted over the particles, which is null in our case.

Then,

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \left(\bar{v}\cdot\bar{\nabla}\right)\Phi + \cancel{\left(\bar{\mathfrak{f}}\cdot\bar{\nabla}_p\right)\Phi} = \left(\frac{\partial\Phi}{\partial t}\right)^{+} - \left(\frac{\partial\Phi}{\partial t}\right)^{-}$$

$\Rightarrow$ Terms of creation and destruction of particles.

The Radiative Transfer Equation

Since $I_\nu(\bar{r}, \bar{n}, t) \equiv \frac{h^4\,\nu^3}{c^2}\,\Phi_\nu(\bar{r}, \bar{n}, t)$

$I_\nu(\bar{r}, \bar{n}, t)$ also satisfies a kinetic equation or Boltzmann equation.

$$\frac{\partial I_\nu}{\partial t} + c\,\left(\bar{n}\cdot\bar{\nabla}\right) I_\nu = \left(\frac{\partial I_\nu}{\partial t}\right)^+ - \left(\frac{\partial I_\nu}{\partial t}\right)^-$$

This equation is usually expressed as a function of the photon path lengths,

$$\boxed{\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \underbrace{\left(\bar{n}\cdot\bar{\nabla}\right)}_{\text{geometry}} I_\nu = \underbrace{\left(\frac{1}{c}\frac{\partial I_\nu}{\partial t}\right)^+ - \left(\frac{1}{c}\frac{\partial I_\nu}{\partial t}\right)^-}_{\text{physical properties of the medium}}}$$

It represents a law of energy conservation.

A rigorous presentation of the RT problem is found in Chandrasekhar (1960), Mihalas (1978), Rybicki and Lightman (1979), Mihalas and Mihalas (1984) and Hubeny and Mihalas (2014).

The Interaction between Matter and Radiation

The interaction between matter and radiation can add or remove radiation along the propagation direction

Absorption of Radiation

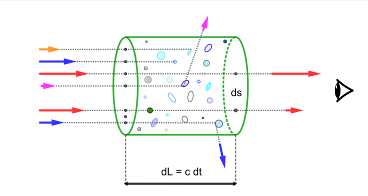


Figure 6: Energy is removed from the beam. Loss of energy due to true absorption-scattering processes.

$[\chi_\nu(\bar{r})]$ in cm^{-1}

Extinction coefficient: $\chi_\nu(\bar{r})$

χ_ν depends on T and P

$$\boxed{\frac{\partial I^-}{\partial L} = \chi_\nu(\bar{r})\, I_\nu(\bar{r}, \bar{n})} \qquad (2)$$

It is composed of:

- true absorption coefficient $\chi_\nu^{th}(T, P)$
- scattering coefficient $\chi_\nu^{s}(T, P)$

$$\boxed{\chi_\nu = \chi_\nu^{th} + \chi_\nu^{s}} \qquad (3)$$

Scattering Processes

The interaction of light with particles (dust, ice) and molecules produces several interesting and important effects

- Refraction
- Reflection
- Scattering
- Diffraction
- Polarisation

Figure 9: Light pillars. Colours and shapes are defined by crystals' location

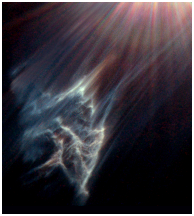
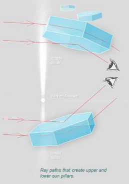


Figure 7: Reflexion nebulae IC 349

Figure 8: Direct reflections from tiny ice crystal. Credits: http://homework.uoregon.edu/pub/class/atm



The scattering process depends on the wavelength to scatter particle size ratio r_s.

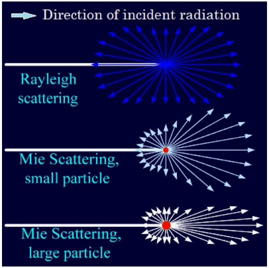
- When particle sizes are tiny relative to the incident wavelength
  $r_s \ll \lambda \Rightarrow$ Rayleigh scattering $\propto \lambda^{-4}$
- When the particle size is comparable or greater than λ
  $r_s \geq \lambda \Rightarrow$ various forms of MIE scattering (spherical particles)
- For large particles, the incoming light is strongly forward scattered.

The MIE theory is essential in studying water droplets, aerosols, dusty disks and the ISM.

Mie theory considers both the scattering and absorption of light by the particle, as well as the angular distribution of scattered light

Blue light (4000 Å) compared to red light (7000 Å) is scattered $(7/4)^4 = 9$ times more efficiently
$\rightarrow$ this explains why the sky is blue!

For cloud water droplets, the wavelengths are scattered in all directions $\rightarrow$ clouds are usually white.



Emission coefficient: $\eta_\nu(\bar{r})$

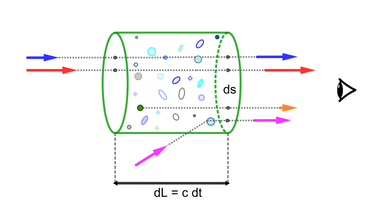


Figure 10: Energy added to the beam due to true absorption and scattering processes.

$[\eta_\nu(\bar{r})]$ is in ergs cm^{-3} s^{-1} Hz^{-1} sr^{-1}

Combine image of the Helix nebulae in Hα, [O III] and [S II] narrow-band filters. Extracted from https://rk.edu.pl/en/helium-argon-and-neon-narrowband-imaging/

Emission coefficient: $\eta_\nu(\bar{r})$

η_ν depends on T and P

$$\frac{\partial I^+}{\partial L} = \eta_\nu(\bar{r})$$

η_ν also depends on T and P It is composed of:

- true emission coefficient $\eta_\nu^{th}(T, P)$
- scattering emission coefficient $\eta_\nu^{s}(T, P)$

$$\eta_\nu = \eta_\nu^{th} + \eta_\nu^{s}$$

The **LTE condition** implies that all small-volume element of a medium is at the **local thermodynamic equilibrium**, so the state of any point can be characterised by the local temperature $T(\bar{r})$.

This supposition is lawful when collisions among particles of the medium occur very frequently. In this case, emissivity and opacity are related by Kirchohoff's law.

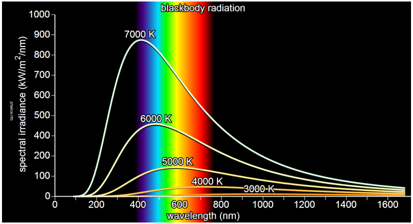


Figure 11: Black body radiation curves at different temperatures .

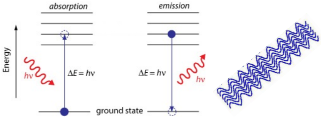
$\frac{\eta_\nu^{th}}{\kappa_\nu^{th}} = B_\nu(T)$ and $B_\nu(T) = \frac{2\,h\,\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$ is the Planck function

Frequency and phase redistribution function

Scattering is a problem in Radiative Transfer codes ⇒ because it strongly modifies the incoming frequency radiation distribution.

Monochromatic scattering

⇒ the photon frequency does not change in the process of their diffusion.

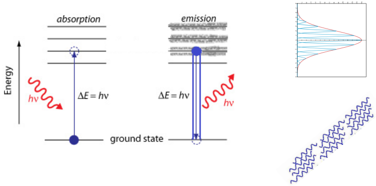


If the atom absorbs a quantum of radiant energy, it must emit a quantum with exactly the same energy if it is to return to its initial state.

This is valid for scattering by large particles and stimulated emission.

Heisenberg's uncertainty principle

The finite life of the upper state implies that the energy of the absorbed and subsequently re-emitted quantum may be different from the energy involved in the electronic transition.



The upper state acts as though it had a finite energy width, permitting the capture of quanta of various frequencies.

Absorption and Re-emission in the same spectrum line

The RT problem is very complicated by the presence of scattering interactions.

These processes redistribute radiation in both frequency and direction and introduce a non-local coupling to the ambient material.

The probability that a photon ν travelling in the direction $\bar{n}$ is absorbed and, then, re-emitted photon with $\nu' = \nu + d\nu$ in the direction n' and $d\Omega'$ is

$$R(\nu, \bar{n}; \nu', \bar{n}') d\nu \, d\Omega_{\bar{n}} \, d\nu' \, d\Omega'_{\bar{n}'}$$

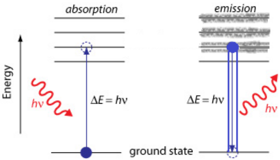
$R(\nu, \bar{n}; \nu', \bar{n}')$ is called the redistribution function

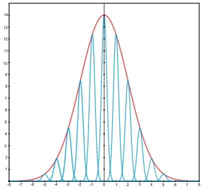
$$\int\int\int\int R(\nu, \bar{n}; \nu', \bar{n}') d\nu \, d\Omega_{\bar{n}} \, d\nu' \, d\Omega'_{\bar{n}'} = 1$$

The absorption profile is defined

$$\phi(\nu) \, d\nu \, d\Omega = 4\pi \, d\nu \, d\Omega \oint \int_{-\infty}^{\infty} R(\nu, \bar{n}; \nu', \bar{n}') \, d\nu' \, d\Omega'_{\bar{n}'}$$

with normalisation $\oint \phi(\nu) \, d\nu \, d\Omega = 4\pi$





If the frequency and phase functions are independent (i.e., in the rest frame), then

$$R(\nu, \bar{n}; \nu', \bar{n}') d\nu\, d\Omega_{\bar{n}}\, d\nu'\, d\Omega'_{\bar{n}'} = p(\nu, \nu')\, g(\bar{n}, \bar{n}')\, d\nu\, d\nu'\, d\Omega_{\bar{n}}\, d\Omega'_{\bar{n}'}$$

The probability that a photon $(\nu', \bar{n}')$ will be absorbed is

$$\chi_{\nu'}(\bar{r})\, I_{\nu'}(\bar{r}, \bar{n}')\, d\nu'\, \frac{d\Omega'_{\bar{n}'}}{4\pi}$$

The probability that the event will be followed by the emission of a photon $(\nu, \bar{n})$ is

$$\eta^s_\nu(\bar{r}) = \int_{-\infty}^{\infty} \oint \chi_{\nu'}(\bar{r})\, I_{\nu'}(\bar{r}, \bar{n}')\, p(\nu, \nu')\, g(\bar{n}, \bar{n}')\, d\nu'\, \frac{d\Omega'_{\bar{n}'}}{4\pi}\, d\nu\, d\Omega_{\bar{n}}$$

- Isotropic scattering $g_A(\bar{n}, \bar{n}') = \frac{1}{4\pi}$
- Dipole scattering $g_B(\bar{n}, \bar{n}') = \frac{3}{16\pi}(1 + \cos^2\gamma)$

The probability that a photon will be emitted is

$$\eta^s_\nu(\bar{r}) = \int_{-\infty}^{\infty} \oint \chi_{\nu'}(\bar{r})\, I_{\nu'}(\bar{r},\bar{n}')\, p(\nu,\nu')\, g(\bar{n},\bar{n}')\, d\nu'\, \frac{d\Omega'_{\bar{n}'}}{4\pi}\, d\nu\, d\Omega_{\bar{n}}$$

For Thomson scattering (grey case)

- Coherence in atom's frame $p(\nu,\nu') = \delta(\nu-\nu')$
- Isotropic scattering $g_A(\bar{n},\bar{n}') = \frac{1}{4\pi}$.

$$\eta^s_\nu(\bar{r}) = \frac{\chi^s}{4\pi} \oint I_\nu(\bar{r},\bar{n}')\, d\Omega'_{\bar{n}'}$$

Defining the mean intensity is

$$J_\nu(\bar{r}) = \frac{1}{4\pi} \oint I_\nu(\bar{r},\bar{n}')\, d\Omega'_{\bar{n}'}$$

$$\boxed{\eta^s_\nu(\bar{r}) = \chi^s J_\nu(\bar{r})}$$

For a Line transition

- Incoherence in atom's frame $p(\nu,\nu') = \phi(\nu-\nu')$
- Isotropic scattering $g_A(\bar{n},\bar{n}') = \frac{1}{4\pi}$.

$$\eta^s_\nu(\bar{r}) = \frac{\chi^s_o}{4\pi} \oint I_\nu(\bar{r},\bar{n}')\, \phi(\nu-\nu')\, d\Omega'_{\bar{n}'}$$

Defining the mean intensity is

$$\bar{J}_\nu(\bar{r}) = \frac{1}{4\pi} \oint I_\nu(\bar{r},\bar{n}')\, \phi(\nu-\nu')\, d\Omega'_{\bar{n}'}$$

$$\boxed{\eta^s_\nu(\bar{r}) = \chi^s \bar{J}_\nu(\bar{r})}$$

The Source Function

Splitting thermal from scattering contributions in the source function, we have

$$S_\nu = \eta_\nu(\bar{r})/\chi_\nu(\bar{r}) = \frac{\eta_\nu^{th} + \eta_\nu^{s}}{\chi_\nu^{th} + \chi_\nu^{s}} = \frac{\eta_\nu^{th}}{\chi_\nu^{th} + \chi_\nu^{s}} + \frac{\eta_\nu^{s}}{\chi_\nu^{th} + \chi_\nu^{s}}$$

Substituting by $\frac{\eta_\nu^{th}}{\chi_\nu^{th}} = B_\nu(T)$ and $\frac{\eta^{s}}{\chi^{s}} = \bar{J}_\nu$, we obtain,

$$S_\nu = \frac{\chi_\nu^{th}}{\chi_\nu^{th} + \chi_\nu^{s}} B_\nu(T) + \frac{\chi_\nu^{s}}{\chi_\nu^{th} + \chi_\nu^{s}} \bar{J}_\nu$$

Then,

$$\boxed{S_\nu = \epsilon_\nu \, B_\nu + \alpha_\nu \, \bar{J}_\nu}$$

Non-local problem

$$\bar{J}_\nu(\bar{r}) = \frac{1}{4\pi} \oint I_\nu(\bar{r}, \bar{n}') \, \phi(\nu - \nu') \, d\Omega'_{\bar{n}'}$$

ϵ_ν and $\alpha_\nu = 1 - \epsilon_\nu$ are the probability of thermal and scattering emission.

If $\Rightarrow \epsilon_\nu = 1 \rightarrow S_\nu = B_\nu(T) \rightarrow$ **Great simplification!**
but not valid for lines!

In mixed cases, RT is an integro-differential equation

$$\boxed{\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \left(\bar{n} \cdot \bar{\nabla}\right) I_\nu = \left(\chi_\nu^{th} + \chi_\nu^{s}\right)\left(S_\nu - I_\nu\right)}$$

Equation of Radiative Transfer

$$\left(\bar{\mathbf{n}}\cdot\bar{\nabla}\right)\mathbf{I}_\nu = \left(\chi_\nu^{th}+\chi_\nu^{s}\right)\left(S_\nu - I_\nu\right)$$

Assuming that $\dfrac{\partial I_\nu}{\partial t} \equiv 0 \rightarrow$ steady state

Spherical Symmetry

$$\frac{dI}{dL} = \frac{\partial}{\partial r} I_\nu(r,\mu)\,\frac{dr}{dL} + \frac{\partial}{\partial\,\theta} I_\nu(r,\mu)\,\frac{d\theta}{dL}$$

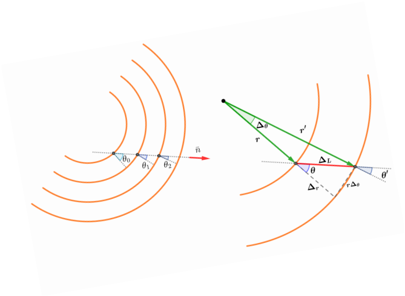
$$dr = dL\cos\theta \quad \rightarrow \quad \frac{dr}{dL} = \cos\,\theta = \mu$$

$$-r\,d\theta = \sin\theta \quad \rightarrow \quad \frac{d\theta}{dL} = -\frac{\sin\,\theta}{r}$$

$(d\theta < 0)$. Then, as $I_\nu(r,\mu)$,

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\mu}\,\frac{\partial\mu}{\partial\theta} = -\sin\theta\,\frac{\partial}{\partial\mu}$$

$$\sin^2\theta = 1-\mu^2$$



$$\Rightarrow \quad \bar{n}\cdot\nabla I_\nu \equiv \mu\,\frac{\partial}{\partial\,r} I_\nu + \frac{1-\mu^2}{r}\,\frac{\partial}{\partial\,\mu} I_\nu$$

$$\boxed{\mu\,\frac{\partial}{\partial r} I_\nu + \frac{1-\mu^2}{r}\,\frac{\partial}{\partial\mu} I_\nu = \kappa_\nu(r)\,\left[\,S_\nu(r) - I_\nu\,\right]}$$

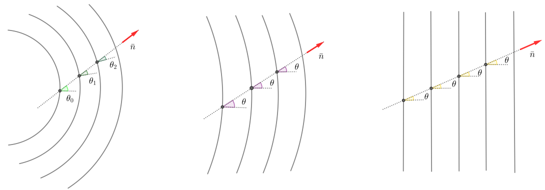


Figure 12: The plane parallel case: when $L/r << 1 \Rightarrow \theta_1 \approx \theta_2$.

Then $\quad \frac{dI}{dL} = \frac{\partial}{\partial r} I_\nu(r,\mu) \, \frac{dr}{dL} + \cancel{\frac{\partial}{\partial \theta} I_\nu(r,\mu)} \, \frac{d\theta}{dL} \quad$ and $\quad \frac{dr}{dL} = \mu$

Thus, the left-hand side term of the RT equation can be approximated by

$$\bar{n} \cdot \bar{\nabla} I_\nu(\bar{r},\mu) \equiv \mu \, \frac{d}{dr} \, I_\nu(r,\mu)$$

and, we obtain,

$$\boxed{\mu \, \frac{d}{dr} I_\nu = \kappa_\nu(r) \, (\, S_\nu(r) - I_\nu \,)} \qquad \text{where} \qquad I_\nu \equiv I_\nu(r,\mu)$$

The Optical Depth

Optical depth measures the attenuation of the radiant power in a medium caused by absorption and scattering processes.

$\tau_\nu = -\int \kappa_\nu(r)\, dr$ is dimensionless

$\tau = 0$ in the outermost atmospheric layer, increasing opposite to r or z coordinate.

- if $\tau_\nu < 1$ optically thin medium
- if $\tau_\nu > 1$ optically thick medium

The photosphere of a star is defined as the surface where its optical depth is $2/3$ (grey atmosphere model). This photospheric border is sometimes adopted at $\tau = 1$ (arbitrary convention).

At $\tau_{5000} = 1, \quad T = T_{\rm eff}, \quad r = R_\star$

Note: the optical depth of a given medium will be different for different **wavelengths** of light.

Thus, the RT equation

$$\mu \frac{d}{dr} I_\nu = \kappa_\nu(r)\ [\, S_\nu(r) - I_\nu \,]$$

can be expressed in terms of the optical depth $d\tau_\nu = -\kappa_\nu(r)\, dr$

$$\boxed{\mu \frac{d}{d\tau_\nu} I_\nu = I_\nu - S_\nu}$$

Formal solution of RT

The formal solution can be obtained using the integrating multiplier method ($e^{-\tau_\nu/\mu}$).

$$\mu\, e^{-\tau_\nu\ \mu} \frac{d}{d\tau_\nu} I_\nu = I_\nu\, e^{-\tau_\nu/\mu} - S_\nu\, e^{-\tau_\nu/\mu}$$

$$\underbrace{e^{-\tau_\nu\ \mu} \frac{d}{d\tau_\nu} I_\nu - \frac{1}{\mu}\, I_\nu\, e^{-\tau_\nu/\mu}} = -S_\nu\, e^{-\tau_\nu/\mu}\, \frac{1}{\mu}$$

$$\frac{d}{d\tau_\nu}\left(I_\nu\, e^{-\tau_\nu/\mu}\right) \quad = \quad - S_\nu\, e^{-\tau_\nu/\mu}\, \frac{1}{\mu}$$

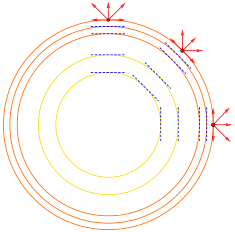


Figure 13: Representation of plane-parallel layers.

Integrating over the optical depth t_ν,

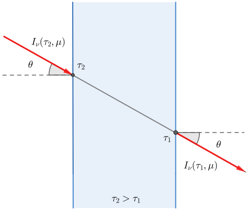
$$d(I_\nu(\tau,\mu)\, e^{-\tau_\nu/\mu})\Big|_{\tau_1}^{\tau_2} = -\int_{\tau_1}^{\tau_2} S_\nu\, e^{-t_\nu/\mu}\, \frac{dt_\nu}{\mu}$$

$$I_\nu(\tau_2,\mu)\, e^{-\tau_2/\mu} - I_\nu(\tau_1,\mu)\, e^{-\tau_1/\mu} = -\int_{\tau_1}^{\tau_2} S_\nu\, e^{-t_\nu/\mu}\, \frac{dt_\nu}{\mu}$$

$$I_\nu(\tau_1,\mu)\, e^{-\tau_1/\mu} = I_\nu(\tau_1,\mu)\, e^{-\tau_2/\mu} + \int_{\tau_1}^{\tau_2} S_\nu\, e^{-t_\nu/\mu}\, \frac{dt_\nu}{\mu}$$

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S_\nu\, e^{-(t_\nu-\tau_1)/\mu}\, \frac{dt_\nu}{\mu}$$

Figure 14: Path of an outgoing light beam in direction $\mu = \cos\theta$ from a layer τ_2 to a less dense layer τ_1.



Then, the complete solution for outgoing radiation at τ_1 is

$$\boxed{I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-\frac{(\tau_2-\tau_1)}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)\, e^{-\frac{(t_\nu-\tau_1)}{\mu}}\, \frac{dt_\nu}{\mu}.}$$

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The formal solution

$$I_\nu(\tau_1,\mu) = \underbrace{I_\nu(\tau_2,\mu)}_{\text{contribution of the radiation from layer } \tau_2} \, e^{\overbrace{-\frac{(\tau_2-\tau_1)}{\mu}}^{\text{attenuation by } \Delta\tau_\nu}} + \int_{\tau_1}^{\tau_2} \underbrace{S_\nu(t_\nu)}_{\text{source function at optical depth } t_\nu} \, e^{\overbrace{-\frac{(t_\nu-\tau_1)}{\mu}}^{\text{attenuation by } (t_\nu - \tau_2)/\mu \text{ (a way to go)}}} \, \frac{dt_\nu}{\mu}$$

This equation is not a real solution because the source function depends on the unknown intensity of the radiation (i.e., J_ν).

$$\boxed{S_\nu = \epsilon_\nu \, B_\nu + \alpha_\nu \, \bar{J}_\nu}$$

We will show later how to solve this problem in static and expanding stellar atmospheres.

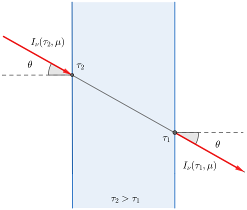


Figure 15: Path of an outgoing light beam in direction $\mu$from a τ_2 layer to a less dense layer at τ_1.

Boundary Conditions

To explicitly solve the RT equation,

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-\dfrac{(\tau_2-\tau_1)}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)\, e^{-\dfrac{(t_\nu-\tau_1)}{\mu}}\, \frac{dt_\nu}{\mu}$$

we need to provide boundary conditions to $I_\nu(\tau_2,\mu)$.

These depend on the physical properties of the layers.

- If boundary layers are transparent or opaque
- If the adjacent surroundings is the vacuum or there is incident radiation

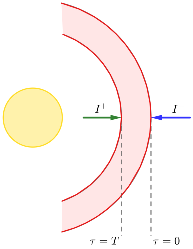
$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-\frac{(\tau_2-\tau_1)}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)\, e^{-\frac{(t_\nu-\tau_1)}{\mu}} \frac{dt_\nu}{\mu}$$

Optical thin medium $\tau << 1$

- For $\tau_1 \equiv 0$ $I_\nu(0,-\mu) \equiv 0$

  no incident radiation
- $\tau_2 = T$ and $I_\nu(T,\mu) = I_\nu^+(\mu)$ (known)

$$I_\nu(0,\mu) = I_\nu^+(\mu)\, e^{-T/\mu} + \int_0^T S_\nu(t_\nu)\, e^{-\frac{t_\nu}{\mu}} \frac{dt_\nu}{\mu}$$



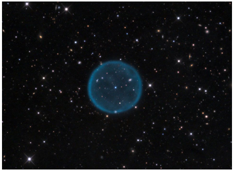


Figure 16: Abel 39

Case of a circumstellar ring or shell.

Optical thick medium $\tau >> 1$

$$I_\nu(\tau_1, \mu) = I_\nu(\tau_2, \mu)\, e^{-\dfrac{(\tau_2 - \tau_1)}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)\, e^{-\dfrac{(t_\nu - \tau_1)}{\mu}} \frac{dt_\nu}{\mu}$$

- $\tau_2 = T$ with $T >> 1$

  $I_\nu(T, \mu) = B_\nu(T)$ or a diffusion equation
- if $\tau_2 \to \infty$

$$\lim_{\tau_\nu \to \infty} I_\nu(\tau_\nu, \mu)\, e^{-\dfrac{(\tau_2 - \tau_1)}{\mu}} = 0$$

(a semi-infinity approximation)

The emergent intensity at $\tau_\nu = 0$ is expressed as

$$\boxed{I_\nu(0, \mu) = \int_0^{\infty} S_\nu(t_\nu)\, e^{-\dfrac{t_\nu}{\mu}} \frac{dt_\nu}{\mu}.}$$

A semi-infinity plane-parallel layer

$$I_\nu(0,\mu) = \int_0^\infty S_\nu(t_\nu)\, e^{-\frac{t_\nu}{\mu}}\, \frac{dt_\nu}{\mu}.$$

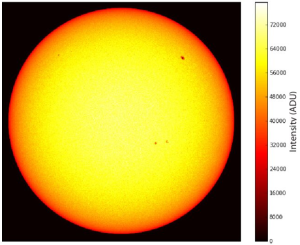


Figure 17: Solar photosphere showing the limb darkening on January 3, 2011 (SDO/HMI takes the solar image

The emergent intensity is the Laplace transform of the source function if the source function is known.

Example: $S_\nu = a\,\tau + b$ then

$I_\nu(0,\mu) = a\,\mu + b$

Limb darkening law

The Momentum of the Radiation Field

$I_\nu(r,\mu)$ contains detailed angular information that can make the radiative transfer problem very hard to solve. Therefore, $I_\nu(r,\mu)$ can expand into spherical harmonics Disadvantage $\rightarrow$ one must specify an axis of reference An equivalent method is to expand into tensor moments For a non-relativistic case, the radiative transfer problem is treated in the scalar moment formalism.

The kth-moment of the radiation field is defined as

$$M_\nu^k(\bar{r}) = \frac{\oint \, I(\bar{r},\bar{n})\,\mu^k\,d\Omega_{\bar{n}}}{\oint \, d\Omega_{\bar{n}}}.$$

In spherical coordinates, the solid angle is

$$d\Omega_{\bar{n}} \, = \, \sin\theta\,d\theta\,d\phi \, = \, -d\mu\,d\phi \qquad \text{and} \qquad \oint d\Omega_{\bar{n}} = 4\,\pi$$

where θ is the co-latitude (from the pole), ϕ is the longitude and $\mu \, = \, cos\theta$.

$$M_\nu^k(\vec{r}) = \frac{1}{4\,\pi}\,\oint I_\nu(\bar{r},\bar{n})\,\mu^n d\Omega_{\bar{n}} \qquad \rightarrow \qquad M_\nu^k(\bar{r}) = \frac{1}{2}\int_{-1}^{1} I_\nu(\bar{r},\mu)\,\mu^n d\mu$$

the first three kth moments of the radiation field, which are scalar functions in spacetime,

For $k = 0$ $\quad M_\nu^o(\bar{r}) = J_\nu$ $\quad$ with $\quad J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\bar{r},\mu)\, d\mu$

$\rightarrow$ Mean Intensity

For $k = 1$ $\quad M_\nu^1(\bar{r}) = H_\nu$ $\quad$ with $\quad H_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\bar{r},\mu)\,\mu\, d\mu$

$\rightarrow$ Eddington flux

For $k = 2$ $\quad M_\nu^2(\bar{r}) = K_\nu$ $\quad$ with $\quad K_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\bar{r},\mu)\,\mu^2\, d\mu$

$\rightarrow$ related with the radiation pressure

The observed flux $F_\nu = 4\, H_\nu$ (the astrophysical flux)

$\mathcal{F}_\nu = 4\,\pi\, H_\nu$ (density flux)

We also use to work with frequency-integrated moments of the distribution function and for the source function, namely

$$M^k \equiv \int_0^\infty M_\nu^k\, d\nu \qquad\qquad S \equiv \int_0^\infty S_\nu\, d\nu$$

Equations of the k^{th} Moments of the Radiation Field

$$M_\nu^k(\bar{r}) = \frac{1}{2}\int_{-1}^{1} I_\nu(\bar{r},\mu)\ \mu^k d\mu$$

$$\mu \frac{\partial}{\partial r} I_\nu(\bar{r},\mu) + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} I_\nu(\bar{r},\mu) = -\chi_\nu \left(I_\nu(\bar{r},\mu) - S_\nu(\bar{r})\right)$$

multiplying both sides by $\frac{1}{2}\,\mu^k$ and integrating over $d\mu$,

The first left-hand side is

$$\frac{1}{2}\int_{-1}^{1}\frac{\partial}{\partial r} I_\nu(\bar{r},\mu)\,\mu^{k+1}\,d\mu = \frac{1}{2}\frac{\partial}{\partial r}\int_{-1}^{1} I_\nu(\bar{r},\mu)\,\mu^{k+1}\,d\mu \equiv \frac{\partial}{\partial r} M_\nu^{k+1}(\bar{r})$$

The second and third terms are

$$\frac{1}{2}\int_{-1}^{1}\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}I_\nu(\bar{r},\mu)\,\mu^k\,d\mu$$

Integrating by parts

$$\frac{1}{2\,r}\int_{-1}^{1}\frac{\partial}{\partial\mu}I_\nu(\bar{r},\mu)\,\mu^k\,d\mu = \frac{1}{2\,r}\,\mu^k I_\nu(\bar{r},\mu)\,d\mu\Big|_{-1}^{1} - \frac{k}{2\,r}\int_{-1}^{1}I_\nu(\bar{r},\mu)\,\mu^{k-1}\,d\mu$$

$$\frac{1}{2\,r}\int_{-1}^{1}\frac{\partial}{\partial\mu}I_\nu(\bar{r},\mu)\,\mu^k\,d\mu = \frac{1}{2\,r}\left[1^{(k)}\,I_\nu(\bar{r},1)-(-1)^k\,I_\nu(\bar{r},-1)\right]-\frac{k}{r}\,M_\nu^{k-1}$$

The third term is very similar

$$\frac{-1}{2\,r}\int_{-1}^{1}\frac{\partial}{\partial\mu}I_\nu(\bar{r},\mu)\,\mu^{k+2}\,d\mu = \frac{1}{2\,r}\left[1^{(k+2)}\,I_\nu(\bar{r},1)-(-1)^{(k+2)}\,I_\nu(\bar{r},-1)\right]-\frac{k+2}{r}\,M_\nu^{k+1}$$

$$\frac{1}{2}\int_{-1}^{1}\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}I_\nu(\bar{r},\mu)\,\mu^k\,d\mu = \frac{k+2}{r}\,M_\nu^{k+1}-\frac{k}{r}\,M_\nu^{k-1}$$

The right-hand side can be written as

$$-\frac{1}{2}\int_{-1}^{1}\chi_\nu(\bar{r})\,I_\nu(\bar{r},\mu)\,\mu^k d\mu+\frac{1}{2}\int_{-1}^{1}\chi_\nu(\bar{r})\,S_\nu(\bar{r})\,\mu^k d\mu=$$

$$-\chi_\nu(\bar{r})M_\nu^k-\frac{1}{2}\int_{-1}^{1}\chi_\nu(\bar{r})\,S_\nu(r)\,\mu^k d\mu$$

Assuming $\chi_\nu(\bar{r})$ is isotropic!

if $k=0$ or k even $\Rightarrow \frac{1}{2}\chi_\nu\,S_\nu(r)\int_{-1}^{1}\mu^k d\mu=\frac{1}{2}\chi_\nu\,S_\nu(r)\,\frac{\mu^{k+1}}{k+1}\Big|_{-1}^{1}$

if k is odd, $\Rightarrow \frac{1}{2}\chi_\nu(\bar{r})\,S_\nu(r)\int_{-1}^{1}\mu^k d\mu=0$

$$\boxed{\frac{\partial}{\partial r}M_\nu^{k+1}+\frac{k+2}{r}M_\nu^{k+1}-\frac{k}{r}M_\nu^{k-1}=-\chi_\nu(\bar{r})\,M_\nu^k+\frac{1}{2}\chi_\nu(\bar{r})\,S_\nu(r)\,\frac{\mu^{k+1}}{k+1}\Big|_{-1}^{1}}$$

The First Equations of Moments

$$\frac{\partial}{\partial r} M_\nu^{k+1} + \frac{k+2}{r} M_\nu^{k+1} - \frac{k}{r} M_\nu^{k-1} = -\chi_\nu M_\nu^k + \frac{1}{2} \chi_\nu S_\nu(r) \frac{\mu^{k+1}}{k+1} \Big|_{-1}^{1}$$

For $k = 0 \rightarrow \frac{\partial H_\nu}{\partial r} + \frac{2}{r} H_\nu = -\chi_\nu J_\nu + \chi_\nu S_\nu(r)$

Integrating over frequencies

$$\frac{1}{r^2} \frac{\partial (r^2 H)}{\partial r} = \int_0^\infty \chi_\nu \left(S_\nu - J_\nu(r)\right) d\nu$$

and assuming radiative equilibrium

$$\int_o^\infty \chi_\nu J_\nu d\nu = \int_o^\infty \chi_\nu S_\nu d\nu$$

$$\bar{\nabla} H = 0$$

and $\chi(r) \neq \chi_\nu(r)$ frequency-independent (grey atmosphere) $\rightarrow$ $\boxed{\mathrm{J} = \mathrm{S}}$

The First Equations of Moments

Spherical Symmetry

- for $k = 0$

$$\frac{1}{r^2}\,\frac{d(r^2 H)}{d\,r} = 0$$

$$H = \frac{\mathcal{C}}{r^2}$$

Flux is diluted!

- for $k = 1$

$$\frac{dK}{dr} + \frac{3\,K - J}{r} = -\chi(r)\,H$$

Plane-parallel case

- for $k = 0$

$$\frac{dH}{dr} = 0$$

$$H = \mathcal{C}$$

The integrated flux is constant

- for $k = 1$

$$\frac{dK}{dr} = -\chi(\bar{r})\,H$$

Comparing the results, the plane-parallel approximation satisfies,

$$\boxed{3\,K = J}$$

The Mean Intensity and Radiation Energy Density

The mean intensity J_ν is the average of the monochromatic intensity over the solid angles,

$$J_\nu(\bar{r}, t) = \frac{1}{4\pi} \oint I_\nu(\bar{r}, \bar{n}, t)\, d\Omega.$$

To understand its physical meaning, it is convenient to calculate the spectral energy density (per unit frequency) of the radiation field.

This spectral energy density is the energy per unit volume $d^3\bar{r}$ and dt,
$\delta E_\nu = I_\nu(\bar{r}, \bar{n}, t)\, d\nu\ ds\, \cos\theta\, dt\, d\bar{\Omega}_{\bar{n}}$

$$U_\nu(r, t) = h\,\nu \oint \frac{h^3 \nu^2}{c^3} \Phi_\nu(r, \mu, t)\, d\Omega_{\bar{n}},$$

Then, using the relation $I_\nu \equiv \frac{h^4\,\nu^3}{c^2}\,\Phi_\nu(\bar{r}, \bar{n}, t)$, we have

$$\boxed{U_\nu(\bar{r}, t) \equiv \frac{1}{c} \oint I_\nu(\bar{r}, \bar{n}, t)\, d\Omega_{\bar{n}} = \frac{4\,\pi}{c}\, J_\nu(\bar{r}, t).}$$

the **density spectral energy**.

Radiative Energy Flux

The energy per unit of time and frequency interval that passes through the surface $ds_{\bar{n}} \perp \bar{n}$ is given by,

$$\frac{\Delta E}{dt\, d\nu} = I_\nu(\bar{r}, \bar{n})\, d\Omega_{\bar{n}}\, ds_{\bar{n}}$$

If the surface is $\perp \bar{n}'$ then

$$\frac{\Delta E}{dt\, d\nu} = I_\nu(\bar{r}, \bar{n})\, d\Omega_{\bar{n}}\, ds_{\bar{n}'}\, cos(\bar{n}, \bar{n}')$$

The flux in the direction $\bar{n}'$ is the sum of the energy that passes through the surface over all directions $\bar{n}$ in the time dt,

$$\bar{F}_{\bar{n}'} = \oint I_\nu(\bar{r}, \bar{n})\, cos(\bar{n}, \bar{n}')\, d\Omega_{\bar{n}}$$

The flux depends on the direction $\bar{n}'$.

If the surface is $\perp \bar{n}$, we have

$$F_x \equiv \oint I_\nu(\bar{r}, \bar{n})\, \mu_x\, d\Omega_{\bar{n}};$$

$$F_y \equiv \oint I_\nu(\bar{r}, \bar{n})\, \mu_y\, d\Omega_{\bar{n}}$$

$$F_z \equiv \oint I_\nu(\bar{r}, \bar{n})\, \mu_z\, d\Omega_{\bar{n}}$$

since $cos(\bar{n}, \bar{n}') = \mu_x\, \mu'_x + \mu_y\, \mu'_y + \mu_z\, \mu'_z$

$$\bar{F}_{\bar{n}'} = F_x\, \mu'_x + F_y\, \mu'_y + F_z\, \mu'_z$$

Transfer of momentum. The Radiation Pressure Tensor

$\bar{F} \cdot d\bar{s}$ is the net rate of radiant energy flow per unit frequency interval, at time dt, across the surface element ds. This represents the pressure exerted on a surface due to the momentum exchange between the radiation field and matter.

Let's compute the components of the moment transported in an arbitrary direction $\bar{n} \equiv (\mu_x, \mu_y, \mu_z)$ through the surface element $\perp$ to each coordinate axis.

$ds_x = ds(1,0,0)$ is a surface element $\perp$ to the x-axis

The energy and momentum of radiation travelling in the direction $\bar{n}$ that passes through the ds_x per unit of time is

$$I_\nu(\bar{r},\bar{n})\, d\Omega_{\bar{n}}\, \mu_x\, ds \qquad \frac{1}{c}\, I_\nu(\bar{r},\bar{n})\, d\Omega_{\bar{n}}\, \mu_x\, ds$$

and their corresponding (x, y, z) components through the surface element ds_x are,

$$\frac{I(\bar{r},\bar{n})\, d\Omega_{\bar{n}}\, ds\, \mu_x\, \mu_x}{c}; \qquad \frac{I(\bar{r},\bar{n})\, d\Omega_{\bar{n}}\, ds\, \mu_x\, \mu_y}{c}; \qquad \frac{I(\bar{r},\bar{n})\, d\Omega_{\bar{n}}\, ds\, \mu_x\, \mu_z}{c}$$

Integrating over all the incoming directions $\bar{n}$, we derive the components of the tensor pressure in the direction x,

$$P_{xx} \equiv \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_x\, \mu_x\, d\Omega_{\bar{n}}; \;\; P_{xy} \equiv \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_x\, \mu_y\, d\Omega_{\bar{n}}; \;\; P_{xz} \equiv \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_x\, \mu_z\, d\Omega_{\bar{n}}$$

Similarly for ds_y and ds_z; we have $(P_{yx},\, P_{yy}\, ,\, P_{yz})$ and $(P_{zx},\, P_{zy}\, ,\, P_{zz})$

$$P_{yx} = \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_y\, \mu_x\, d\Omega_{\bar{n}}; \;\; P_{yy} = \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_y\, \mu_y\, d\Omega_{\bar{n}}; \;\; P_{yz} = \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_y\, \mu_z\, d\Omega_{\bar{n}}$$

$$P_{zx} = \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_z\, \mu_x\, d\Omega_{\bar{n}}; \;\; P_{zy} = \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_z\, \mu_y\, d\Omega_{\bar{n}}; \;\; P_{zz} = \frac{1}{c} \oint I(\bar{r}, \bar{n})\, \mu_z\, \mu_z\, d\Omega_{\bar{n}}$$

This way, the pressure tensor,

$$\overline{\overline{P}} = \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix}$$

where $P_{xy} = P_{yx}$, $P_{xz} = P_{zx}$, and $P_{yz} = P_{zy}$.

The Scalar Pressure

$$P_{ij} = \frac{1}{c} \oint I(\bar{r},\bar{n})\, \mu_i\, \mu_j\, d\Omega_{\bar{n}}$$

$$\text{for example} \rightarrow P_{zz} = \frac{1}{c} \oint I(\bar{r},\bar{n})\, \mu_z^2\, d\Omega_{\bar{n}}$$

$$P_{\rm rad} = \frac{1}{3\,c} \oint (P_{xx} + P_{yy} + P_{zz})\, d\Omega_{\vec{n}} = \frac{1}{3\,c} \oint I(\vec{r},\vec{n})\, d\Omega_{\vec{n}}$$

$$\boxed{P_{\rm rad} = \frac{4\,\pi}{3\,c}\, J}$$

The Radiation Force

The negative gradient of the pressure tensor is the radiation force per unit volume. The x-component of this force $\mathbb{F}(\bar{n})$ is given by

$$\mathbb{F}_x = -(\bar{\nabla}\overline{\overline{P}})_x = -\bar{\nabla}(P_{xx}, P_{xy}, P_{xz})$$

$$-(\bar{\nabla}\overline{\overline{P}})_x = -\frac{1}{c}\oint\left(\frac{\partial I(\bar{r},\bar{n})}{\partial x}\,\mu_x\,\mu_x + \frac{\partial\, I(\bar{r},\bar{n})}{\partial\, y}\,\mu_x\,\mu_y + \frac{\partial I(\bar{r},\bar{n})}{\partial\, z}\,\mu_x\,\mu_z\right)\, d\Omega_{\bar{n}}$$

$$\boxed{\mathbb{F}(\bar{n})_x = \frac{1}{c}\oint\, \mu_x\,\chi_\nu\,(I_\nu - S_\nu)\, d\Omega_{\bar{n}}}$$

Similarly, for the other two components of the force,

$$\boxed{\mathbb{F}(\bar{n})_y = \frac{1}{c}\oint\, \mu_y\,\chi_\nu\,(I_\nu - S_\nu)\, d\Omega_{\bar{n}}}$$

$$\boxed{\mathbb{F}(\bar{n})_z = \frac{1}{c}\oint\, \mu_z\,\chi_\nu\,(I_\nu - S_\nu)\, d\Omega_{\bar{n}}}$$

Physics of Extreme Massive Stars
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Summary

$$J_\nu(\bar{r}) = \frac{1}{4\,\pi} \oint I_\nu(\bar{r}, \bar{n})\, d\Omega_{\bar{n}}$$

Mean intensity

$$U_\nu(\bar{r}, t) = \frac{\text{radiation energy}}{\text{volume}} = \frac{4\,\pi}{c} J_\nu(\bar{r}, t)$$

Energy density

$$P_\nu(\bar{r}) = \frac{1}{c} \oint I_\nu(\bar{r}, \bar{n})\, \mu^2\, d\Omega_{\bar{n}}$$

Radiation pressure

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{d\,\text{momentum}(E/c)}{dt} \frac{1}{\text{area}}$$

$$F(\bar{n})_x = \frac{1}{c} \oint \mu_x\, \chi_\nu\, (I_\nu - S_\nu)\, d\Omega_{\bar{n}}$$

Radiation force

It depends on the density flux $F = C/r^2$, $\chi(\bar{r})$ and the orientation of the surface with respect to force.

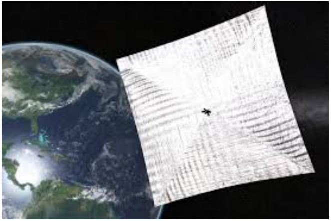
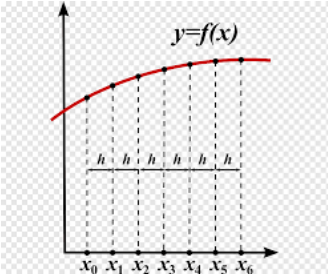


Figure 18: LightSail 2 spacecraft, launched on June 25, 2019, and reentered Earth's atmosphere on Nov. 17, 2022. It used sunlight alone to change its orbit.

Numerical Methods

- Integral methods
- Statistical methods
- Finite difference methods
- Finite element methods
- Mixed methods



The Feautrier Method

Example for the plane parallel case: $\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$

In the $0 \leq \mu \leq 1$ we have two first order equations

$$\pm\mu \frac{dI\left(\tau_\nu, \pm\mu, \nu\right)}{d\tau_\nu} = I\left(\tau_\nu, \pm\mu, \nu\right) - S\left(\tau_\nu, \nu\right)$$

We define the symmetric and anti-symmetric variables of the radiation field:

$$U\left(\tau_\nu, \mu, \nu\right) = \frac{1}{2}\left[I\left(\tau_\nu, \mu, \nu\right) + I\left(\tau_\nu, -\mu, \nu\right)\right]$$

$$V\left(\tau_\nu, \mu, \nu\right) = \frac{1}{2}\left[I\left(\tau_\nu, \mu, \nu\right) - I\left(\tau_\nu, -\mu, \nu\right)\right]$$

Combining both equations

$$\mu \frac{dV_\nu}{d\tau_\nu} = U_\nu - S_\nu$$

$$\mu \frac{dU_\nu}{d\tau_\nu} = V_\nu$$

Eliminating V_ν, we obtain a second order differential equation

$$\boxed{\mu^2 \frac{d^2 U_\nu}{d\tau_\nu^2} = U_\nu - S_\nu}$$

where

$$S(\tau_\nu, \nu) = \alpha J(_\nu, \nu) + \beta$$

$$J(\tau_\nu, \nu) = \frac{1}{2}\int_0^1 I(\tau_\nu, -\mu, \nu)\, d\mu + \frac{1}{2}\int_0^1 I(\tau_\nu, \mu, \nu)\, d\mu \equiv \int_0^1 U(\tau_\nu, \mu, \nu)\, d\mu$$

Now, the source function is in terms of $U(\tau_\nu, \mu, \nu)$

$$S(\tau_\nu, \nu) = \alpha \int_0^1 U(\tau_\nu, \mu, \nu)\, d\mu + \beta$$

$$\boxed{\mu^2 \frac{d^2 U_\nu}{d\tau_\nu^2} = U_\nu - \alpha \int_0^1 U(\tau_\nu, \mu, \nu)\, d\mu + \beta}$$

Boundary conditions

From first-order equations:

$$\mu \frac{dV_\nu}{d\tau_\nu} = U_\nu - S_\nu$$

$$\mu \frac{dU_\nu}{d\tau_\nu} = V_\nu$$

Outer border

$$I\left(0, -\mu, \nu\right) \equiv 0 \quad \Rightarrow \quad U\left(0, \mu, \nu\right) = V\left(0, \mu, \nu\right)$$

$$\Rightarrow \quad \mu \left.\frac{dU_\nu}{d\tau_\nu}\right|_{\tau_{\text{min}}} = U\left(\tau_{\text{min}}, \mu, \nu\right).$$

Inner border $\tau_\nu = \tau_{\text{max}}$

$I^+ = I\left(\tau_{\text{max}}, \mu, \nu\right)$ i.e. $B_\nu(T)$

$$U\left(\tau_{\text{max}}, \mu, \nu\right) + V\left(\tau_{\text{max}}, \mu, \nu\right) = I^+\left(\tau_{\text{max}}, \mu, \nu\right)$$

$$\mu \left.\frac{dU_\nu}{d\tau_\nu}\right|_{\tau_{\text{max}}} = I^+ - U\left(\tau_{\text{max}}, \mu, \nu\right).$$

Finite Difference Equations

$$S\left(\tau_\nu,\nu\right)=\alpha J\left(_\nu,\nu\right)+\beta$$

$$J_d=\sum_{i=1}^{4}w_i U_d\left(\mu_i\right)=\sum_{i=1}^{4}w_i U_{di}$$

$$S_d=\alpha\sum_{i=1}^{4}w_i U_{di}+\beta=\alpha\left(w_1 U_{d1}+w_2 U_{d2}+w_3 U_{d3}+w_4 U_{d4}\right)+\beta$$

Building the grids

- $\tau=\left(\tau_1,\tau_2,\tau_3,\ldots,\tau_{d-1},\tau_d,\tau_{d+1},\ldots,\tau_D\right)$
- $\mu_d=\left(\mu_1,\mu_2,\ldots,\mu_m,\ldots,\mu_M\right)$
- $\nu_d=\left(\nu_1,\nu_2,\ldots,\nu_n,\ldots,\nu_N\right)$

The first derivative is evaluated at $\tau_{d-1/2}$

$$\left.\frac{dU}{d\tau}\right|_{d+1/2} \approx \left.\frac{\Delta U}{\Delta \tau}\right|_{d+1/2} = \frac{\Delta U_{d+1/2}}{\Delta \tau_{d+1/2}} = \frac{U_{d+1} - U_d}{\tau_{d+1} - \tau_d}.$$

$$\left.\frac{dU}{d\tau}\right|_{d-1/2} \approx \left.\frac{\Delta U}{\Delta \tau}\right|_{d-1/2} = \frac{\Delta U_{d-1/2}}{\Delta \tau_{d-1/2}} = \frac{U_d - U_{d-1}}{\tau_d - \tau_{d-1}}$$

Instead, the second derivative is defined on the grid.

$$\frac{\mu_i^2}{\Delta \tau_d}\left[\frac{U_{d+1,i} - U_{d,i}}{\Delta \tau_{d+1/2}} - \frac{U_{d,i} - U_{d-1,i}}{\Delta \tau_{d-1/2}}\right] = U_{d,i} - \alpha \sum_j w_j \, U_{d,j} - \beta$$

Then, we have

$$-\left(-\frac{\mu_i^2}{\Delta\tau_d \Delta\tau_{d-1/2}}\right) U_{d-1,i}$$
$$+\left(-\frac{\mu_i^2}{\Delta\tau_d \Delta\tau_{d+1/2}} - \frac{\mu_i^2}{\Delta\tau_d \Delta\tau_{d-1/2}} + \sum_j \alpha w_j - 1\right) U_{d,i}$$
$$-\left(-\frac{\mu_i^2}{\Delta\tau_d \Delta\tau_{d+1/2}}\right) U_{d+1,i} = -\beta$$

$$-A_d \, \overline{U_{d-1}} + B_d \, \overline{U_d} - C_d \, \overline{U_{d+1}} = \overline{L_d}$$

$$\overline{U_d} = \begin{pmatrix} U_{d1} \\ U_{d2} \\ U_{d3} \\ U_{d4} \end{pmatrix}$$

The same can be done for the boundary conditions.

Examples of some matrix arragements

$$A_d = \begin{pmatrix} -\frac{\mu_1^2}{\Delta\tau_d \Delta\tau_{d-1/2}} & 0 & 0 & 0 \\ 0 & -\frac{\mu_2^2}{\Delta\tau_d \Delta\tau_{d-1/2}} & 0 & 0 \\ 0 & 0 & -\frac{\mu_3^2}{\Delta\tau_d \Delta\tau_{d-1/2}} & 0 \\ 0 & 0 & 0 & -\frac{\mu_4^2}{\Delta\tau_d \Delta\tau_{d-1/2}} \end{pmatrix}$$

$$\mathrm{B}_d = \begin{pmatrix} B_1^* + \alpha w_1 & \alpha w_2 & \alpha w_3 & \alpha w_4 \\ \alpha w_1 & B_2^* + \alpha w_2 & \alpha w_3 & \alpha w_4 \\ \alpha w_1 & \alpha w_2 & B_3^* + \alpha w_3 & \alpha w_4 \\ \alpha w_1 & \alpha w_2 & \alpha w_3 & B_4^* + \alpha w_4 \end{pmatrix}$$

Boundary conditions

$$A_1 \equiv 0 \qquad C_D \equiv 0$$

Forward Elimination and Backsubstitution

$$-A_d\,\overline{U_{d-1}} + B_d\,\overline{U_d} - C_d\,\overline{U_{d+1}} = \overline{L_d}$$

Border conditions

$$A_1 \equiv 0 \qquad\qquad C_D \equiv 0$$

$$B_1\,\overline{U_1} - C_1\,\overline{U_2} = \overline{L_1}$$

$$\overline{U_1} = D_1\,\overline{U_2} + \overline{V_1} \qquad \left\{\begin{array}{l} D_1 = B_1^{-1}\,C_1 \\ \overline{V_1} = B_1^{-1}\,\overline{L_1} \end{array}\right.$$

$$\overline{U_d} = D_d\,\overline{U_{d+1}} + \overline{V_d}$$

where

$$\left\{\begin{array}{l} D_d = [B_d - A_d\,D_{d-1}]^{-1}\;C_d \\ \overline{V_d} = [B_d - A_d\,D_{d-1}]^{-1}\left[\overline{L_d} + A_d\,\overline{V_{d-1}}\right] \end{array}\right.$$

From the inner boundary condition $\overline{U_D} \equiv \overline{V_D}$
Substituting in

$$\overline{U_d} = D_d\,\overline{U_{d+1}} + \overline{V_d}$$

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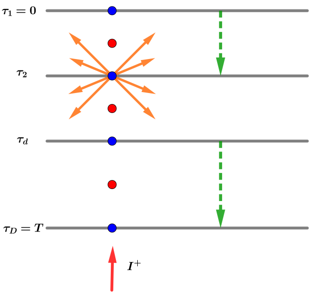
Feautrier Standard Solution Scheme



- Define a grid of depth (τ), angle (μ) and frequency (*nu*)
- Depth by depth, define the matrix elements A, B, C and keep the values
- Go forward layer by layer
- Use the inner boundary conditions
- Use the recurrent equation to compute
  $\overline{U_d} = D_d \, \overline{U_{d+1}} + \overline{V_d}$
- Compute $U(\tau, \nu)$ and the source function.

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Behaviour of the Source Function

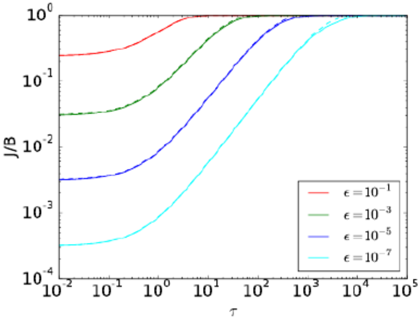

Figure 4. J computed from HERO (dashed lines) compared with the analytic solution (solid lines) for a plane parallel scattering atmosphere described under the 2-stream approximation. Four values of ϵ (see Eq. 7) are shown.