Discontinuous Finite Element Method in Moving Media

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Abstract. We describe the solution of the one-dimensional radiative transfer equation in moving media using the discontinuous finite element method. First, we apply the static solution to the case of a nonzero gradient of velocities using the Lorentz transformation. Then, we solve more exactly the equation of the radiative transfer including aberration and without it. We test these methods for a case of a B4 main sequence star with a stellar wind. All three techniques give comparable results.

1. Introduction

In past, several classes of methods for a solution of the radiative transfer equation (hereafter RTE) have been used. In this paper we deal with a method of discontinuous finite elements (hereafter DFE). This method was first introduced into radiative transfer by Castor et al. (1992). Further improvements and generalization of this method to more spatial dimensions were published by Dykema et al. (1996). Recently, a finite element method was applied by Richling et al. (2001) to the solution of the monochromatic static RTE in three dimensions.

2. The method of discontinuous finite elements

The DFE method is a generalization of the finite element method, which has been used very often in various calculations in physics in last few decades. We briefly describe the essence of the DFE method here.

We section the given region to the cells. In every cell we consider the solution to be a linear combination of solutions at grid points and some base functions in the Hilbert space of quadratically integrable functions. This approximation of a solution we insert to the differential equation being solved, multiply it by base functions and integrate over the zone. We obtain a system of algebraic equations for grid values of unknowns. The jump is included by adding a solution obtained in a previous cell and subtracting the solution from the current cell.

In the next section, we describe the application of this method to the solution of the RTE in moving atmospheres.
3. Solution of the radiative transfer equation in moving media

3.1. Solution using the local Lorenz transformation

We adopt the method of a solution of the one-dimensional static plane-parallel RTE
\[ \frac{dI(z, \mu, \nu)}{dz} = \eta(z, \nu) - \chi(z, \nu)I(z, \mu, \nu) \] (1)
from Castor et al. (1992) and apply their method to moving media.

Since the DFE method enables inclusion of a jump in the intensity at the cell boundary, the treatment of a velocity field is very simple. In every cell we assume a constant velocity of a medium and allow a change of the velocity only at the boundary of the cells. Since the RTE is Lorentz invariant, we can solve the static solution inside the cell and perform the Lorentz transformation of the intensity and frequency at a boundary. We call this method the Local Lorentz Transformation (LLT) method.

This method is the most effective in a case when the velocity gradient is not “too large”. We simply have to guarantee that the sets of new and old frequency points at the cell boundary significantly overlap. With an increasing velocity gradient the number of cells has also to increase and, consequently, one needs more computing time to solve the equation. However, for “very” large velocity gradients we may use the Sobolev method. Using the method of the Local Lorentz Transformation it is possible to solve the RTE also for nonmonotonic velocity fields. We found that for the case of a stellar wind this method yields satisfactory results.

3.2. The direct solution of the RTE in moving media

In addition to the simple method of a Local Lorentz Transformation, the DFE method may be also applied directly to the solution of the RTE in the comoving frame. If we neglect aberration, the comoving frame RTE is
\[ \mu_0 \frac{\partial I_0(z, \mu_0, \nu_0)}{\partial z} - \frac{\mu_0^2 \nu_0}{c} \frac{\partial \nu_0}{\partial z} \frac{\partial I_0(z, \mu_0, \nu_0)}{\partial \nu_0} = \eta_0(z, \nu_0) - \chi_0(z, \nu_0)I_0(z, \mu_0, \nu_0) \] (2)

We discretize the geometrical depth \( z = \{z_d\}, d = 1, \ldots, D \) and the frequency \( \nu_0 = \{\nu_{0,n}\}, n = 1, \ldots, N \). In every cell we place a plane, which passes through the intensity grid points of the cell. The intensity inside the cell is expressed as a linear combination of intensities in grid points and space functions,
\[ I_{z, \nu_0} = \sum_{k=1}^{4} w_k I_k, \] (3)
where the intensities \( I_k \) and the base functions \( w_k \) are
\[ I_1 = I_{d,n} \quad I_2 = I_{d+1,n} \quad I_3 = I_{d,n+1} \quad I_4 = I_{d+1,n+1} \] (4)
\[ w_1 = \frac{z_{d+1} - z}{z_{d+1} - z_d} \frac{\nu_{0,n+1} - \nu_0}{\nu_{0,n+1} - \nu_{0,n}} \quad w_2 = \frac{z - z_d}{z_{d+1} - z_d} \frac{\nu_{0,n+1} - \nu_0}{\nu_{0,n+1} - \nu_{0,n}} \]
\[ w_3 = \frac{z_{d+1} - z}{z_{d+1} - z_d} \frac{\nu_0 - \nu_{0,n}}{\nu_{0,n+1} - \nu_{0,n}} \quad w_4 = \frac{z - z_d}{z_{d+1} - z_d} \frac{\nu_0 - \nu_{0,n}}{\nu_{0,n+1} - \nu_{0,n}} \] (5)
We insert the expression for the intensity (3) to the RTE (2), multiply it by the base function (5), and integrate over the zone. We obtain a system of algebraic equations

$$\mathbf{A} \mathbf{I} = \mathbf{b}. \quad (6)$$

We assume that all terms in the Eq. (2) change as the intensity within a cell, i.e.

$$\eta(z, \nu) = \sum_{k=1}^{4} w_k \eta_k \quad \chi(z, \nu) I(z, \nu) = \sum_{k=1}^{4} w_k \chi_k I_k \quad (7)$$

$\eta_k$ and $\chi_k$ are introduced similarly to the intensity (4). We add the jump of the intensity (see the left panel of the Figure 1)

$$\begin{pmatrix}
I_1 - (I_{d,n,2} + I_{d,n,3} + I_{d,n,4})/3 \\
0 \\
I_3 - (I_{d,n+1,2} + I_{d,n+1,4})/2 \\
0
\end{pmatrix} \quad (8)$$

to the system of equations (6), which we solve for every cell. The resulting intensity in a grid point $I_{d,n}$ is an arithmetic average of the intensities $I_k$ ($I_{d,n} = \frac{1}{4} \sum_{k=1}^{4} I_k$). Since our method solves the differential equation of the first order, to obtain the information about the whole radiation field we must solve the equation (2) both in upward and downward directions with corresponding boundary conditions.

If we do not neglect aberration, we have to solve the equation

$$\left[ \left( \mu_0 + \frac{v}{c} \frac{\partial}{\partial z} + \frac{\mu_0 (\mu_0^2 - 1)}{c} \frac{\partial v}{\partial z} \frac{1}{\partial \mu_0} - \frac{v_0 \mu_0^2}{c} \frac{\partial v}{\partial \nu_0} \frac{\partial}{\partial \nu_0} \right) + 3 \frac{\mu_0^2}{c} \frac{\partial}{\partial z} \right] I_0(z, \mu_0, \nu_0) = \eta_0(z, \nu_0) - \chi_0(z, \nu_0) I_0(z, \mu_0, \nu_0), \quad (9)$$

where $\eta_0$ and $\chi_0$ are the terms of the intensity (4).
which results in a system of eight algebraic equations. Consequently, the calculations will be more time consuming.

3.3. Test calculations

We performed our test calculations for an atmosphere of a B4 star. A corresponding LTE model atmosphere has been calculated using a model atmosphere code ATA (see Kubát, these proceedings). For the dynamic case we adopted for the velocity field the beta law (Cassinelli & Lamers 1999) with photospheric velocity of 200 km · s\(^{-1}\) and terminal velocity of 2000 km · s\(^{-1}\). The results of our computation are shown in the right panel of the Fig. 1.

The reliability of the DFE method is supported by the fact that in a static one-dimensional case the solution of the RTE using the DFE method agrees well with the classical Feautrier method.

4. Conclusions

For the case of media with velocity gradients we developed two classes of applications of the DFE method to the solution of the RTE. The first class solves the RTE directly as a differential equation in \(z\) and \(\nu\) (and \(\mu\) for the case of aberration), the second class use the static solution with local Lorentz Transformation. Results from the latter method agree with the more exact direct solution of the RTE. Comparison of the solution with the inclusion of the effect of aberration and without it showed only a small difference, in accordance with older results. So, for nonrelativistic velocity fields it is better to calculate without aberration, because the computing time is significantly shorter. Basic advantage of the DFE method is the possibility of solving the RTE also for nonmonotonic velocity fields. On the other hand, we found that this method is a bit unstable in some cases, especially in the case of small optical depths. We also found that the DFE method converged to a wrong solution when an optical depth was used as an independent variable instead of \(z\). The reason of this is in the numerical calculation of the system of algebraic equations. While the geometrical depth does not change very much throughout the atmosphere, the optical depth changes by several orders of magnitude. Despite the above mentioned problems, the DFE method is a powerful possibility of solution of the RTE. We plan to employ our results to future studies of Be stars.

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References