Cool white dwarf model atmospheres and astrophysical applications

• Motivation:

1 Model atmospheres are used to determine $T_{\text{eff}}$, $\log g$, and $\log \text{He/H}$, $\log \text{C/H}$, etc.

2 Last year, I introduced the the quasi-static line broadening theory applied to hydrogen line profiles. In particular I showed that in the line wings the line opacity is well described by $\alpha(\Delta \nu) \propto \Delta \nu^{-5/2}$ and that hydrogen lines are useful temperature/pressure diagnostics.

3 With these ($T_{\text{eff}}$, $\log g$) diagnostics, one may infer the age and distance of white dwarf stars using theoretical cooling tracks: large samples of white dwarf stars may then be used to verify the validity of the input physics and retrace the evolution of white dwarf stars.

4 The development of model atmospheres for cool white dwarf stars require proper attention to convective energy transports and molecular opacities.

• These notes are based in part on work done by Adela Kawka and Eva Arazimova and on the Principles of Stellar Evolution and Nucleosynthesis by D.D. Clayton.
To introduce the subject let’s look at our favorite star:

Figure 1: An optical/infrared spectrum of the Sun, a blackbody at 5770K and a white dwarf model at the same temperature, but log \( g \) = 8 (compared to log \( g_{\odot} \) = 4.44).

(1) Much of the optical and infrared resembles a blackbody... and so does the spectrum of a white dwarf.

(2) But note the spectral lines ... and the effect of H\(^-\)(bf) opacity near 1.5\( \mu \)m discovered by Chandrasekhar & Breen (1946)!
• Can we really use model atmospheres and produce a white dwarf HR diagram like this one?

Figure 2: A white dwarf HR diagram. Curves of constant masses are displayed at 0.4 (top), 0.6, 0.8, and 1.0 $M_\odot$ (bottom).

(1) The diagram shows absolute V magnitudes for samples of hot EUV-selected white dwarfs (Vennes et al.) and cool high-proper motion white dwarfs (Kawka, Arazimova, et al.).
(2) To determine $M_V$ we follow the logical sequence:

$$(T_{\text{eff}}, \log g) \rightarrow (M/M_\odot, R/R_\odot, \text{age}) \rightarrow (T_{\text{eff}}, M_V)$$

(a) Balmer line fits (see last year and below) ...

(b) Mass-radius relations (Chandrasekhar; Hamada & Salpeter 1961; Wood 1995; Benvenuto & Althaus 1999) ...

(c) And a spectral synthesis valid over the photometric bandpass.

Figure 3: A white dwarf Balmer line fit (Arazimova).

(3) But to produce these models we had to introduce \textit{convective energy transport} following the classical \textit{mixing length theory}.
- Convective energy transport.

Figure 4: A stellar convection zone.

→ The energy carrier in the convection process is labelled an “eddy” or a cell. In stars, eddies consist of a gas regions expanding adiabatically: it does not not exchange energy with the surrounding.

Let’s work in the pressure-volume space. An alternative to specific volume is the density. Since we work with pressure and volume the adiabatic exponent to work with is $\Gamma_1$. 
We assume pressure equilibrium ... \( P^* \) inside the cell
= \( P \) in the ambiant medium.

But the specific density (or volume) of the cell and ambiant medium are not necessarily equal!

(a) Inside the adiabatic cell we have (\( V = \rho^{-1} \)):

\[
\frac{dP}{P} = -\Gamma_1 \left( \frac{dV}{V} \right)_* = \Gamma_1 \left( \frac{d\rho}{\rho} \right)_*
\]

where the “*” subscript refers to the cell.

At the base of the convection region the density of the cell is identical to the density of the ambiant medium:

\( \rho^*(r) = \rho(r) \)

But after moving up by \( dr \) the density inside the cell is given by:

\[
\rho^*(r + dr) = \rho + d\rho^* = \rho + \frac{1}{\Gamma_1} \frac{\rho \, dP}{P \, dr} \, dr
\]

What is the buoyancy of the cell?

- cell is stable if: \( \rho^*(r + dr) > \rho(r + dr) \)
- cell is unstable if: \( \rho^*(r + dr) < \rho(r + dr) \)

that is the gas is stable if:

\[
\left( \frac{d\rho}{dr} \right)_* = \frac{1}{\Gamma_1} \frac{\rho \, dP}{P \, dr} > \frac{d\rho}{dr}
\]
(2) It is more appropriate to work with the temperature gradient (use adiabatic exponent $\Gamma_2$ when working with $P$ and $T$). And in a very similar fashion the gas is stable if:

$$-\left(\frac{dT}{dr}\right)_* = -\frac{(\Gamma_2 - 1) T}{\Gamma_2} \frac{dP}{P} dr > -\frac{dT}{dr}$$

Because $(dT/dr)$ is negative, the criterion suggest that if the temperature varies more rapidly in the medium than inside the cell, instability arises and convection sets in.

In more compact notation:

$$\text{cell is stable if: } -\nabla_{\text{ad}} > -\nabla_{\text{actual}}$$

(3) How much energy is carried inside a cell? Assuming the gas is unstable, the cell has an excess temperature at the end of its motion relative to ambient medium:

$$dT = (\nabla_{\text{ad}} - \nabla_{\text{actual}}) dr = \Delta(\nabla T) dr$$

$dT$ is positive since $-\nabla_{\text{ad}} < -\nabla_{\text{actual}}$ is true. Read the last equation delta-nabla-T.
(4) Following a simple dimensional analysis, the convective flux is given by:
\[ H = (\Delta(\nabla T)dr) \ C_P \rho v \]
where the *first* factor is the temperature difference, the *second* factor tells how much energy per K is stored, the *third* factor tells how much of the gas is involved, the *fourth* factor tells how fast the cell is carrying that energy upward.

(5) The cell is accelerated by buoyant force due to the density difference between the cell and the ambiant medium over a distance \( l \). At the start of the motion \( \Delta \rho = 0 \) and the end of the motion:
\[ \Delta \rho = l \left[ \left( \frac{\partial \rho}{\partial r} \right) - \left( \frac{\partial \rho}{\partial r} \right)_* \right] = l \Delta \nabla \rho \]
Applying trapezoidal rule for the average of the density difference over the entire path:
\[ < \Delta \rho > = \frac{1}{2} l \Delta \nabla \rho \]
and the average buoyant acceleration is:
\[ a = \frac{< \Delta \rho >}{\rho} g = \frac{1}{2} \frac{g l}{\rho} \Delta \nabla \rho \]
And in a simple dynamical world $v_j^2 = v_i^2 + 2al = 2al$
we have applying again the trapezoidal rule:

$$< v > = \frac{v_f}{2} = \frac{1}{2} \sqrt{\frac{g l^2}{\rho} \Delta \nabla \rho}$$

(6) We still have to evaluate $\Delta \nabla \rho$. Since $\rho \propto P/T$:

$$\frac{1}{\rho} \frac{d\rho}{dr} = \frac{1}{P} \frac{dP}{dr} - \frac{1}{T} \frac{dT}{dr}$$

and for the adiabatic expansion:

$$\left( \frac{d\rho}{dr} \right)_* = \frac{1}{\Gamma_1} \frac{\rho}{P} \frac{dP}{dr}$$

We stated that:

$$\Delta \nabla \rho = \left[ \left( \frac{\partial \rho}{\partial r} \right) - \left( \frac{\partial \rho}{\partial r} \right)_* \right]$$

Then:

$$\Delta \nabla \rho = \left[ \frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{T} \frac{dT}{dr} - \frac{1}{\Gamma_1} \frac{\rho}{P} \frac{dP}{dr} \right]$$

$$\Delta \nabla \rho = \frac{\rho}{T} \left[ \frac{T}{P} \frac{dP}{dr} - \frac{dT}{dr} - \frac{1}{\Gamma_1} \frac{T}{P} \frac{dP}{dr} \right]$$

And assume that $\Gamma_1 \approx \Gamma_2$ (quite close actually):

$$\Delta \nabla \rho = \frac{\rho}{T} \left[ - \frac{dT}{dr} - \frac{1 - \Gamma_2}{\Gamma_2} \frac{T}{P} \frac{dP}{dr} \right]$$
Now use the second form of the adiabatic expansion:

\[ \frac{dP}{P} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0 \quad \text{2nd form} \]

and rewrite:

\[ \left( \frac{dT}{dr} \right)_* = -\frac{1 - \Gamma_2 T}{\Gamma_2} \frac{dP}{P} \frac{dr}{dr} \]

\[ \Delta \nabla \rho = \frac{\rho}{T} \left[ -\frac{dT}{dr} + \left( \frac{dT}{dr} \right)_* \right] = \frac{\rho}{T} (\nabla_{\text{ad}} - \nabla_{\text{actual}}) = \frac{\rho}{T} \Delta \nabla T \]

So the mean velocity is now expressed as a function of the temperature gradients:

\[ <v> = \frac{1}{2} \sqrt{\frac{g l^2 \rho}{\rho T}} \Delta \nabla T = \frac{1}{2} \sqrt{\frac{g}{T}} \Delta \nabla T \]

and taking the convective flux over \( dr \sim l \) where \( l \) is the mixing-length, we have for the flux

\[ H = (\Delta(\nabla T)dr) \ C_P \ \rho \ v = \Delta(\nabla T) \ l \ C_P \ \rho \frac{l}{2} \sqrt{\frac{g}{T}} \Delta \nabla T \]

and simplify

\[ H_{\text{conv}} = [\Delta(\nabla T)]^{3/2} C_P \ \frac{l^2}{2} \sqrt{\frac{g}{T}} \]

where \( l \) is parametrized as a factor of the pressure scale height.
This expression for the convective flux must be incorporated in the expression for the total flux:

\[ 4\pi H = 4\pi H_{\text{conv}} + 4\pi \int H_\nu d\nu = \sigma T_{\text{eff}}^4 \]

where we modify the equation of radiative equilibrium to incorporate the convective flux contribution. Such equations may be solved using the complete linearization method.

In computing a simple grey atmosphere (achromatic opacities) it is sufficient to consider these equations.

<table>
<thead>
<tr>
<th>hydrostatic equilibrium:</th>
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<tbody>
<tr>
<td>[ \frac{dP}{dr} = -\frac{G , M(r)}{r^2} \rho ]</td>
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<tr>
<th>radiative transfer:</th>
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<td>[ \frac{dT}{dr} = -\frac{L_r}{4\pi r^2} \left( \frac{4acT^3}{3\bar{\chi}_R} \right)^{-1} ]</td>
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<td>[ \left( \frac{dT}{dr} \right)_* = -\frac{1 - \Gamma_2 T}{\Gamma_2} \frac{dP}{P} ]</td>
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• The heat capacities $C_P$ and $C_V$ play an important role in convective energy transport:

$$C_V = \left( \frac{dQ}{dT} \right)_V \quad C_P = \left( \frac{dQ}{dT} \right)_P$$

and for an ideal monoatomic gas:

$$C_V = \frac{dU}{dT} = \frac{d}{dT} \left( \frac{3}{2} nRT \right) = \frac{3}{2} nR$$

And employing the ideal gas law it follows that

$$C_P = \frac{dQ}{dT} = \frac{dU}{dT} + P \frac{dV}{dT} = C_V + nR = \frac{5}{2} nR$$

These relations for heat capacities are only valid for mono atomic gas such as a completely ionized gas ... or completely neutral atomic gas.

$$C_V = \frac{3}{2} nR \quad C_P = C_V + nR$$

The adiabatic exponent is defined as:

$$\gamma = \frac{C_P}{C_V}$$

which for a perfect mono-atomic gas is:

$$\gamma = \frac{C_P}{C_V} = \frac{5/2}{3/2} = \frac{5}{3}$$

... or $7/5$ for diatomic gas and $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma$. 
OPEN A PARENTHESES ...

For an ideal gas:

\[ PV^\gamma = \text{constant adiabatic} \]

Using \( PV = nRT \) you can in turn show that:

\[ TV^{\gamma - 1} = \text{constant adiabatic} \]

\[ P^{1-\gamma}T^\gamma = \text{constant adiabatic} \]

For monoatomic ideal gas, the gas exponent \( \gamma \) applies to all three forms of the adiabatic expansion.

For non-ideal gas, subjected to inelastic collisions, they are not equal in general. Chandrasekhar defined the three adiabatic exponents:

\[ PV^{\Gamma_1} = \text{constant, or } \frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0 \quad 1^{st} \text{ form} \]

\[ P^{1-\Gamma_2}T^{\Gamma_2} = \text{constant, or } \frac{dP}{P} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0 \quad 2^{nd} \text{ form} \]

\[ TV^{\Gamma_3 - 1} = \text{constant, or } \frac{dT}{T} + (\Gamma_3 - 1) \frac{dV}{V} = 0 \quad 3^{rd} \text{ form} \]

... AND CLOSE THAT PARENTHESES)
(1) The effect of ionization on the adiabatic expansion and compression of a real gas.

For an ideal gas the same exponent $\gamma = C_P/C_V$ governs adiabatic expansion and compression in the $(P, V, T)$ space. For monoatomic ideal gas, the adiabatic surface is uniquely characterized by $\gamma = 5/3$.

For real gas show deviation from this simple behavior. Consider a gas comprising neutral hydrogen atoms, protons, and electrons:

$$n^0 + \chi \leftrightarrow n^+ + e^-$$

where $\chi$ is the ionization potential in the above reaction. The possibility of ionization adds a term to the internal energy:

$$U = uV = \frac{3}{2}NVkT + N^+V\chi$$

that is each proton potentially carries a excess of $\chi = 13.6$ eV. Here, $N = N^0 + N^+ + N_e = N^0 + 2N^+$ is the total number of particles. Therefore, by definition

$$C_V = \left(\frac{dU}{dT}\right)_V = \frac{3}{2}NVk + \frac{3}{2}VkT\left(\frac{dN}{dT}\right)_V + \chi V\left(\frac{dN^+}{dT}\right)_V$$

and you noted that $N$ is no longer a constant! $N$ depends on the number protons that trade-off that energy $\chi$ for an electron.
We need to work out some of the ingredients required...

... first the volume is constant and equal to:

\[ V^{-1} = \rho \equiv M_u(N^0 + N^+) \]

where we neglected the contribution of electrons to density. Taking the derivative with respect to temperature at constant \( V = \rho^{-1} \):

\[ \left( \frac{dN^+}{dT} \right)_V = -\left( \frac{dN^0}{dT} \right)_V \]

and because the total density is given by \( N = N^0 + 2N^+ \) we have as well:

\[ \left( \frac{dN}{dT} \right)_V = -\left( \frac{dN^0}{dT} \right)_V \]

and assembling these results we have for the heat capacity:

\[ C_V = \frac{3}{2} NVk + \frac{3}{2} V kT \left( \frac{dN}{dT} \right)_V + \chi V \left( \frac{dN^+}{dT} \right)_V \]

\[ C_V = \frac{3}{2} NVk - \left( \frac{3}{2} V kT + \chi V \right) \left( \frac{dN^0}{dT} \right)_V \]

and factorizing \( 3NkV/2 \):

\[ C_V = \frac{3}{2} NVk \left[ 1 - \frac{2T}{3N} \left( \frac{3}{2} + \frac{\chi}{kT} \right) \left( \frac{dN^0}{dT} \right)_V \right] \]

It appears that \( C_V \) is equal to its value for an ideal mono-atomic gas plus a correction term.
This is a good place to introduce the Saha equation which describes the equilibrium particle density:

\[
\frac{N^0}{N^+ N_e} = \left( \frac{\hbar^2}{2\pi m_e kT} \right)^{3/2} e^{\chi/kT}
\]

Note the effect of temperature, and the effect of particle density on the direction a reaction is likely to evolve. Because in the present situation \(N^+ = N_e\):

\[
N^0 = (N^+)^2 \Phi(T)
\]

Calculating the derivatives using the Saha equation is a lot of fun and defining the ionization fraction:

\[
y = \frac{N^+}{(N^+ + N^0)}
\]

you can show that:

\[
C_V = \frac{3}{2} NV k \left[ 1 + \frac{2}{3} \left( \frac{3}{2} + \frac{\chi}{kT} \right)^2 \frac{(1 - y) y}{(1 + y)(2 - y)} \right]
\]

\[
C_V = \frac{3}{2 \mu M_u} k \left[ 1 + \frac{2}{3} \left( \frac{3}{2} + \frac{\chi}{kT} \right)^2 \frac{(1 - y) y}{(1 + y)(2 - y)} \right]
\]

And we also show that:

\[
C_P = \frac{3}{2 \mu M_u} k \left[ \frac{5}{3} + \frac{1}{3} \left( \frac{5}{2} + \frac{\chi}{kT} \right)^2 (1 - y) y \right]
\]

using \(PV = nRT\).
In this calculation $\chi = 13.6$ eV is the ionization potential of hydrogen, and we adopted $T = 10000$ K which is typical of partional ionization zone of hydrogen.

Note that $C_V$ increases by a factor of two between $y = 0$ and $y = 1$ because the number of particles per gram is effectively doubled.
The adiabatic gradient in a partial ionization zone varies from $5/3 = 1.667$ to $\approx 1.14$ ... 

Other reactions alter the ideal gas picture as well:

$$n_l^0 + E_{lu} \leftrightarrow n_u^0$$

$$n(H_2) + \chi_{H_2} \leftrightarrow n^0 + n^0$$

where $\chi_{H_2} = 4.5$ eV is the molecular hydrogen dissociation energy...

$$n(H^-) + \chi_{H^-} \leftrightarrow e^- + n^0$$

where $\chi_{H^-} = 0.75$ eV ...
• Sample model atmosphere results:

(1) Opacities in cool atmospheres

\[ T_{\text{eff}} = 5600, \log(g) = 8 \]

Note the collision-induced absorption (CIA) when two \( \text{H}_2 \) molecules form a super-molecule with a temporary (weak) dipole moment. Similar reaction occurs between \( \text{H}_2 \) and He.

Figure 7: \( \chi \) versus wavelength in \( \mu \text{m} \).
(2) Temperature and pressure structures in cool atmospheres.

Figure 8: $P$ and $T$ versus Rosseland depth ($\tau_R$).
(3) Temperature gradients in cool atmospheres:

\[ \nabla \] versus Rosseland depth \((\tau_R)\).

Figure 9: \( \nabla \) versus Rosseland depth \((\tau_R)\).
(4) Populations in cool atmospheres:

Figure 10: Population fractions versus pressure.
(5) Populations in cool atmospheres:

Figure 11: Population fractions versus pressure.
• Spectral synthesis production:

(1) NLTT 2286 (CTIO July 2007, 2MASS) as a possible H-He mixed atmosphere. Shows evidence of H$_2$-He collision-induced absorption.

Figure 12: Spectral energy distribution of NLTT 2286.
(2) Or the possibly pure hydrogen NLTT 8432 (CTIO July 2007, 2MASS),

![Figure 13: Spectral energy distribution of NLTT 8432.](image-url)
and NLTT 33669 (CTIO July 2007, 2MASS).

Figure 14: Spectral energy distribution of NLTT 33669.

$T = 6000 \text{ K}$
(4) Finally, Balmer line profile analysis of a massive white dwarf:

Figure 15: A white dwarf Balmer line fit (Arazimova).
- Population properties based on Balmer line profile analyses.

Figure 16: Measured log $g$ versus $T_{\text{eff}}$.
The measurements are compared to theoretical mass-radius relations of Wood (1995) at 0.4 (bottom), 0.6, 0.8, 1.0 and 1.2 $M_\odot$ (top) with cooling ages labelled.
which we converted into masses and ages:

\[(T_{\text{eff}}, \log g) \rightarrow (M/M_\odot, R/R_\odot, \text{age})\]

Figure 17: Measured mass versus age.

Note the population of ultramassive white dwarfs at \(M > 1.1M_\odot\).
... and finally into the HR diagram:

$$(T_{\text{eff}}, \log g) \rightarrow (M/M_\odot, R/R_\odot, \text{age}) \rightarrow (T_{\text{eff}}, M_V)$$

Figure 18: A white dwarf HR diagram. Curves of constant masses are displayed at 0.4 (top), 0.6, 0.8, and 1.0 $M_\odot$ (bottom).
• White dwarf kinematics:

\[ \mu_\alpha, \mu_\delta, v_{\text{radial}}, \text{distance} \rightarrow U, V, W \]

Figure 19: U versus V galactic velocity vectors for the NLTT sample of high proper-motion white dwarfs (Arazimova).

Note the halo locus which lags by \( \approx 200 \text{ km s}^{-1} \), and the thick/thin disk loci centered on the local standard of rest (LSR).
Galactic orbit of peculiar and ordinary white dwarfs ... following the Galactic potential $\Phi$ model of Flynn et al. (1996, MNRAS 281, 1027) and coded by Flynn.

(1) The halo white dwarf par-excellence PMJ 13420$-$3415 (Lepine et al.), the post-binary LP400-22 (Kawka et al.), and two typical disk white dwarfs NLTT 7051 and NLTT 8432:

Figure 20: Galactic orbit in the $Z$ versus $R$ plane.
(2) and zoomed onto the two disk white dwarfs. Note the slight eccentricity of NLTT 8432.

Figure 21: Galactic orbit in the Z versus R plane.

The cooling age of NLTT 7051 is 2.9 Gyr and the cooling age of NLTT 8432 is 1.5 Gyr. But NLTT 7051 (0.54$$M_\odot$$) possibly had a more massive shorter-lived progenitor than NLTT 8432 (0.50$$M_\odot$$). The total age of NLTT 8432 may be longer which, through multiple encounters, may have increased its eccentricity.
(3) And viewed from above the plane, the eccentric orbit of NLTT 8432, the circular orbit of NLTT 7051, the Y-static halo orbit of PMJ 13420–3415, and the wild excursions of the post-binary LP400-22 ...

Figure 22: Galactic orbit in the X versus Y plane.
• What are my conclusions?

(1) \[(T_{\text{eff}}, \log g) \rightarrow (M/M_\odot, R/R_\odot, \text{age}) \rightarrow (T_{\text{eff}}, M_V) \rightarrow \text{(distance)}\]

(2) \[\mu_\alpha, \mu_\delta, v_{\text{radial}}, \text{distance} \rightarrow U, V, W\]

(3) Kinematics and age tell the full history of the white dwarf population.

(4) Cool white dwarf model atmospheres are difficult to build.